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CHARACTERISTIC FUNCTIONS AND JOINT INVARIANT SUBSPACES

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Abstract. Let $T := [T_1, \ldots, T_n]$ be an n-tuple of operators on a Hilbert space such that T is a completely non-coisometric row contraction. We establish the existence of a "one-to-one" correspondence between the joint invariant subspaces under $T_1, \ldots, T_n$, and the regular factorizations of the characteristic function Θ_T associated with T. In particular, we prove that there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$, if and only if there is a non-trivial regular factorization of Θ_T. We also provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function, and prove the existence of joint invariant subspaces for certain classes of n-tuples of operators.

We obtain criterions for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we prove that a completely non-coisometric row contraction T is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator.

1. Introduction

In the classical case of a single operator, the connection between the invariant subspaces of an operator and the corresponding characteristic function was first considered, for certain particular classes of operators, in the work of Livšitz, Potapov, Šmulyan, Brodsky, etc (see the references from [22] and [23]). One of the fundamental results in the Nagy-Foiaş theory of contractions [25] states that the invariant subspaces of a completely non-unitary (c.n.u.) contraction T on a (separable) Hilbert space are in "one-to-one" correspondence with the regular factorizations of the characteristic function associated with T. This general result, although influenced in part by the work of the authors cited above, was obtained by Sz.-Nagy and Foiaş in [22], [23], following an entirely different approach based on the geometric structure of the unitary dilation and the corresponding functional model for c.n.u. contractions.

The main goal of this paper is to obtain a multivariable version of the above-mentioned result, for n-tuples of operators, and to provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function. This comes as a natural continuation of our program to develop a *free* analogue of Nagy-Foiaş theory, for row contractions.

An n-tuple $T := [T_1, \ldots, T_n]$ of bounded linear operators acting on a common Hilbert space $\mathcal{H}$ is called *row contraction* if

$$T_1T_1^* + \cdots + T_nT_n^* \leq I.$$

A distinguished role among row contractions is played by the n-tuple $S := [S_1, \ldots, S_n]$ of *left creation operators* on the full Fock space with n generators, $F^2(H_n)$, which satisfies the

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noncommutative von Neumann inequality [10] (see also [12], [14])

$$\|p(T_1, \ldots, T_n)\| \leq \|p(S_1, \ldots, S_n)\|$$

for any polynomial $p(X_1, \ldots, X_n)$ in n noncommuting indeterminates. For the classical von Neumann inequality [26] (case $n = 1$) and a nice survey, we refer to Pisier's book [5]. Based on the left creation operators and their representations, a noncommutative dilation theory and model theory for row contractions was developed in [3], [2], [6], [7], [8], [11], etc. In this study, the role of the unilateral shift is played by the left creation operators and the Hardy algebra $H^\infty(\mathbb{D})$ is replaced by the noncommutative analytic Toeplitz algebra F_n^∞. We recall that F_n^∞ was introduced in [10] as the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^*-closed) algebra generated by the left creation operators $S_1, \ldots, S_n$ and the identity.

In [8], we defined the *standard characteristic function* of a row contraction (a multi-analytic operator acting on Fock spaces) which, as in the classical case ($n = 1$) [25], turned out to be a complete unitary invariant for completely non-coisometric row contractions (c.n.c.). We also constructed a model for c.n.c. row contractions, in which the characteristic function occurs explicitely.

In 2000, Arveson [1] introduced and studied the curvature and Euler characteristic associated with a row contraction with commuting entries. Noncommutative analogues of these numerical invariants were defined and studied by the author [15] and, independently, by D. Kribs [4]. We showed in [19] that the curvature invariant and Euler characteristic asssociated with a Hilbert module generated by an arbitrary (resp. commuting) row contraction $T := [T_1, \ldots, T_n]$ can be expressed only in terms of the (resp. constrained) characteristic function of T.

In this paper, we continue the study of the characteristic function Θ_T associated with a row contraction $T := [T_1, \ldots, T_n]$ in connection with joint invariant subspaces under the operators $T_1, \ldots, T_n$, and the joint similarity of T to a Cuntz row isometry $W := [W_1, \ldots, W_n]$, i.e., $W_1, \ldots, W_n$ are isometries with

$$W_1 W_n^* + \cdots + W_n W_n^* = I.$$

After some preliminaries on multivariable noncommutative dilation theory (see Section 2), we present in Section 3 the main results of this paper.

We establish the existence of a "one-to-one" correspondence between the joint invariant subspaces under $T_1, \ldots, T_n$, and the regular factorizations of the characteristic function Θ_T associated with a completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$ (see Theorem 3.2 and Theorem 3.6). In particular, we prove that there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$, if and only if there is a non-trivial regular factorization of Θ_T (see Theorem 3.7). Using the model theory for c.n.c. row contractions, we provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function (see Theorem 3.3). An important question related to the main result, Theorem 3.2, is to what extent a joint invariant subspace determines the corresponding regular factorization of the characteristic function. We address this problem in Theorem 3.8.

In Section 4, we prove the existence of a unique triangulation of type

$$\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix}$$

for any row contraction $T := [T_1, \ldots, T_n]$ (see Theorem 4.1), and prove the existence of non-trivial joint invariant subspaces for certain classes of row contractions. We also show

that there is a non-trivial joint invariant subspace under $T_1, \ldots, T_n$ whenever the inner-outer factorization of the characteristic function associated with T is non-trivial (see Theorem 4.6).

In Section 5, we obtain criterions for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we prove that a completely non-coisometric row contraction T is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator (see Theorem 5.2). Moreover, in this case, we provide a model Cuntz row isometry for similarity. This is a multivariable version of a result of Sz.-Nagy and Foiaş [24], concerning the similarity to unitary operators.

Extending on some results obtained by Sz.-Nagy [21], Nagy-Foiaş [25], and the author [6], [17], we prove, in particular, that a one-to-one power bounded n-tuple $[T_1, \ldots, T_n]$ of operators on a Hilbert space $\mathcal{H}$ is jointly similar to a Cuntz row isometry if and only if there exists a constant $c > 0$ such that

$$\sum_{\alpha \in \mathbb{F}_n^+, |\alpha| = k} \|T_\alpha^* h\|^2 \geq c\|h\|^2, \quad h \in \mathcal{H},$$

for any $k = 1, 2, \ldots$.

Recently [19], [20] we developed a dilation theory for row contractions $[T_1, \ldots, T_n]$ subject to constraints such as $p(T_1, \ldots, T_n) = 0$, $p \in \mathcal{P}$, where $\mathcal{P}$ is a set of noncommutative polynomials. It would be interesting to see to what extent the results of this paper can be extended to *constrained row contractions* and their *constrained characteristic functions*.

2. Preliminaries on characteristic functions for row contractions

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$ or $n = \infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n),\ i = 1, \ldots, n$, by

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n).$$

The noncommutative analytic Toeplitz algebra F_n^∞ and its norm closed version, the noncommutative disc algebra $\mathcal{A}_n$, were introduced by the author [10] in connection with a multivariable noncommutative von Neumann inequality. F_n^∞ is the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^*-closed) algebra generated by the left creation operators $S_1, \ldots, S_n$ acting on $F^2(H_n)$, and the identity. When $n = 1$, F_1^∞ can be identified with $H^\infty(\mathbb{D})$, the algebra of bounded analytic functions on the open unit disc. The algebra F_n^∞ can be viewed as a multivariable noncommutative analogue of $H^\infty(\mathbb{D})$. There are many analogies with the invariant subspaces of the unilateral shift on $H^2(\mathbb{D})$, inner-outer factorizations, analytic operators, Toeplitz operators, $H^\infty(\mathbb{D})$–functional calculus, bounded (resp. spectral) interpolation, etc.

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$, and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := k$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| := 0$, if $\alpha = g_0$. We also define $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ and $e_{g_0} = 1$. It is clear that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $F^2(H_n)$. If $T_1, \ldots, T_n \in B(\mathcal{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, we define $T_\alpha := T_{i_1} T_{i_2} \cdots T_{i_k}$ and $T_{g_0} := I_{\mathcal{H}}$.

We need to recall from [8], [9], [10], [12], and [13] a few facts concerning multi-analytic operators on Fock spaces. We say that a bounded linear operator A acting from $F^2(H_n)\otimes \mathcal{K}$ to $F^2(H_n)\otimes \mathcal{K}'$ is multi-analytic if

$$A(S_i\otimes I_{\mathcal{K}})=(S_i\otimes I_{\mathcal{K}'})A \quad \text{for any} \quad i=1,\dots,n. \tag{2.1}$$

Notice that A is uniquely determined by the operator $\theta:\mathcal{K}\to F^2(H_n)\otimes\mathcal{K}'$, which is defined by $\theta k:=A(1\otimes k)$, $k\in\mathcal{K}$, and is called the symbol of A. We denote $A=A_\theta$. Moreover, A_θ is uniquely determined by the "coefficients" $\theta_{(\alpha)}\in B(\mathcal{K},\mathcal{K}')$, which are given by

$$\left\langle \theta_{(\tilde\alpha)}k,k'\right\rangle := \left\langle \theta k, e_\alpha\otimes k'\right\rangle = \left\langle A_\theta(1\otimes k), e_\alpha\otimes k'\right\rangle, \quad k\in\mathcal{K},\ k'\in\mathcal{K}',\ \alpha\in\mathbb{F}_n^+,$$

where $\tilde\alpha$ is the reverse of α, i.e., $\tilde\alpha=g_{i_k}\cdots g_{i_1}$ if $\alpha=g_{i_1}\cdots g_{i_k}$. We can associate with A_θ a unique formal Fourier expansion

$$A_\theta\sim\sum_{\alpha\in\mathbb{F}_n^+} R_\alpha\otimes\theta_{(\alpha)},$$

where $R_i:=U^*S_iU$, $i=1,\dots,n$, are the right creation operators on $F^2(H_n)$ and U is the unitary operator on $F^2(H_n)$ mapping $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}$ into $e_{i_k}\otimes\cdots\otimes e_{i_2}\otimes e_{i_1}$. Based on the noncommutative von Neumann inequality [12], we proved that

$$A_\theta=\text{SOT}-\lim_{r\to 1}\sum_{k=0}^\infty\sum_{|\alpha|=k} r^{|\alpha|}R_\alpha\otimes\theta_{(\alpha)},$$

where, for each $r\in(0,1)$ the series converges in the uniform norm. The set of all multi-analytic operators in $B(F^2(H_n)\otimes\mathcal{K},F^2(H_n)\otimes\mathcal{K}')$ coincides with $R_n^\infty\bar\otimes B(\mathcal{K},\mathcal{K}')$, the WOT closed algebra generated by the spatial tensor product, where $R_n^\infty:=U^*F_n^\infty U$ (see [13] and [16]). The multi-analytic operator A_θ is called

(i) inner if A_θ is an isometry,
(ii) outer if $\overline{A_\theta(F^2(H_n)\otimes\mathcal{E})}=F^2(H_n)\otimes\mathcal{E}_*$,
(iii) purely contractive if $\|P_{\mathcal{E}_*}\theta h\|<\|h\|$ for every $h\in\mathcal{E}$, $h\neq 0$,
(iv) unitary constant if $A_\theta=I\otimes W$ for some unitary operator $W\in B(\mathcal{K},\mathcal{K}')$.

If $A_{\theta'}: F^2(H_n)\otimes\mathcal{E}'\to F^2(H_n)\otimes\mathcal{E}_*'$ is another multi-analytic operator, we say that A_θ coincides with $A_{\theta'}$ if there exist two unitary operators

$$W:\mathcal{E}\to\mathcal{E}', \quad W_*:\mathcal{E}_*\to\mathcal{E}_*'$$

such that

$$(I\otimes W_*)A_\theta=A_{\theta'}(I\otimes W).$$

For simplicity, throughout this paper, $T:=[T_1,\dots,T_n]$, $n=1,\dots,\infty$, denotes either the n-tuple $(T_1,\dots,T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ or the row operator matrix $[T_1\ \cdots\ T_n]$ acting from $\mathcal{H}^{(n)}$ to $\mathcal{H}$, where $\mathcal{H}^{(n)}:=\oplus_{i=1}^n\mathcal{H}$ is the direct sum of n copies of $\mathcal{H}$. Assume that $T:=[T_1,\dots,T_n]$ is a row contraction, i.e.,

$$T_1T_1^*+\cdots+T_nT_n^*\leq I.$$

The defect operators of T are

$$\Delta_{T^*}:=\left(I_{\mathcal{H}}-\sum_{i=1}^n T_iT_i^*\right)^{1/2}\in B(\mathcal{H}) \quad\text{and}\quad \Delta_T:=(I_{\mathcal{H}^{(n)}}-T^*T)^{1/2}\in B(\mathcal{H}^{(n)}),$$

and the defect spaces of T are defined by

$$\mathcal{D}_*:=\overline{\Delta_{T^*}\mathcal{H}} \quad\text{and}\quad \mathcal{D}:=\overline{\Delta_T\mathcal{H}^{(n)}}.$$

The characteristic function of the row contraction $T := [T_1, \dots, T_n]$ is the multi-analytic operator $\Theta_T : F^2(H_n) \otimes \mathcal{D} \to F^2(H_n) \otimes \mathcal{D}_*$ with symbol θ_T is given by

$$\theta_T(h) := -\sum_{i=1}^n T_i P_i h + \sum_{i=1}^n (S_i \otimes I_{\mathcal{D}_*}) \left(\sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes \Delta_{T^*} T_\alpha^* P_i \Delta_T h \right), \qquad h \in \mathcal{D},$$

where P_i denotes the orthogonal projection of $\mathcal{H}^{(n)}$ onto the i-component of $\mathcal{H}^{(n)}$, and $S := [S_1, \dots, S_n]$ is the model multi-shift of left creation operators acting on the full Fock space $F^2(H_n)$.

Using the characterization of multi-analytic operators on Fock spaces (see [13], [16]), one can easily see that the characteristic function of T is a multi-analytic operator with the formal Fourier representation

$$-I \otimes T + (I \otimes \Delta_{T^*}) \left(I - \sum_{i=1}^n R_i \otimes T_i^* \right)^{-1} [R_1 \otimes I_{\mathcal{H}}, \dots, R_n \otimes I_{\mathcal{H}}] \, (I \otimes \Delta_T),$$

where $R_1, \dots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$.

The definition of the characteristic function of T arises in a natural way in the context of the theory of noncommutative isometric dilations for row contractions (see [7] and [8]). Let $V := [V_1, \dots, V_n]$, $V_i \in B(\mathcal{K})$, be the minimal isometric dilation of T on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. Therefore,

(i) $V_1, \dots, V_n$ are isometries with orthogonal ranges;
(ii) $T_i^* = V_i^*|_{\mathcal{H}}$, $i = 1, \dots, n$;
(iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

Consider the following subspaces of $\mathcal{K}$:

$$\mathcal{L} := \bigvee_{i=1}^n (V_i - T_i)\mathcal{H}, \qquad \mathcal{L}_* := \overline{\left(I_{\mathcal{K}} - \sum_{i=1}^n V_i T_i^* \right) \mathcal{H}}.$$

According to [7], we have the following orthogonal decompositions of the minimal isometric dilation space of T:

$$\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*) = \mathcal{H} \oplus M_V(\mathcal{L}), \tag{2.2}$$

where $\mathcal{R}$ reduces each operator V_i, $i = 1, \dots, n$,

$$M_V(\mathcal{L}_*) = \bigoplus_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{L}_*, \quad \text{and} \quad M_V(\mathcal{L}) = \bigoplus_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{L}.$$

Denote by $\Phi^{\mathcal{L}}$ the unitary operator from $M_V(\mathcal{L})$ to $F^2(H_n) \otimes \mathcal{L}$ defined by

$$\Phi^{\mathcal{L}} \left(\sum_{\alpha \in \mathbb{F}_n^+} V_\alpha \ell_\alpha \right) := \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes \ell_\alpha, \qquad \ell_\alpha \in \mathcal{L}, \ \sum_{\alpha \in \mathbb{F}_n^+} \|\ell_\alpha\|^2 < \infty.$$

One can view $\Phi^{\mathcal{L}}$ as the Fourier representation of $M_V(\mathcal{L})$ on Fock spaces. Then, for any $i = 1, \dots, n$, we have

$$\Phi^{\mathcal{L}} V_i = (S_i \otimes I_{\mathcal{L}}) \Phi^{\mathcal{L}},$$

where $S := [S_1, \dots, S_n]$ is the model multi-shift of left creation operators acting on the full Fock space $F^2(H_n)$. Similarly, one can define the unitary operator (Fourier representation)

$\Phi^{\mathcal{L}_*}: M_V(\mathcal{L}_*) \to F^2(H_n)\otimes\mathcal{L}_*$. We proved in [8] that the characteristic function Θ_T coincides with the multi-analytic operator $\Theta_{\mathcal{L}}: F^2(H_n)\otimes\mathcal{L} \to F^2(H_n)\otimes\mathcal{L}_*$ defined by

$$\Theta_{\mathcal{L}} := \Phi^{\mathcal{L}_*}(P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{L})})(\Phi^{\mathcal{L}})^*,$$

where $P_{M_V(\mathcal{L}_*)}$ denotes the orthogonal projection of $\mathcal{K}$ onto $M_V(\mathcal{L}_*)$.

Let $T := [T_1, \dots, T_n]$, $n = 1, \dots, \infty$, be a row contraction with $T_i \in B(\mathcal{H})$ and consider the subspace $\mathcal{H}_c \subset \mathcal{H}$ defined by

$$\mathcal{H}_c := \left\{ h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \text{ for any } k = 1, 2, \dots \right\}$$

We call T a completely non-coisometric (c.n.c.) row contraction if $\mathcal{H}_c = \{0\}$. We proved in [7] that $\mathcal{H}_c$ is a joint invariant subspaces under the operators $T_1^*, \dots, T_n^*$, and it is also the largest subspace in $\mathcal{H}$ on which T^* acts isometrically. Consequently, we have the following triangulation with respect to the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{cnc}$:

$$T_i = \begin{pmatrix} A_i & 0 \\ * & B_i \end{pmatrix}, \qquad i = 1, \dots, n,$$

where $[A_1, \dots, A_n]$ is a coisometry, i.e., $A_1A_1^* + \cdots + A_nA_n^* = I_{\mathcal{H}_c}$, and $[B_1, \dots, B_n]$ is a c.n.c. row contraction.

In [8], we constructed the following model for c.n.c. row contractions, in which the characteristic function occurs explicitly.

Theorem 2.1. *Every completely non-coisometric row contraction $T := [T_1, \dots, T_n]$, $n = 1, 2, \dots, \infty$, on a Hilbert space $\mathcal{H}$ is unitarily equivalent to a row contraction $\mathbf{T} := [\mathbf{T_1}, \dots, \mathbf{T_n}]$ on the Hilbert space*

$$\mathbf{H} := [(F^2(H_n)\otimes\mathcal{D}_*) \oplus \overline{\Delta_{\Theta_T}(F^2(H_n)\otimes\mathcal{D})}] \ominus \{\Theta_T f \oplus \Delta_{\Theta_T} f : \quad f \in F^2(H_n)\otimes\mathcal{D}\},$$

where $\Delta_{\Theta_T} := (I - \Theta_T^*\Theta_T)^{1/2}$ and the operator $\mathbf{T}_i$, $i = 1, \dots, n$, is defined by

$$\mathbf{T}_i^*[f \oplus \Delta_{\Theta_T}(S_j\otimes I_{\mathcal{D}_*})g] := \begin{cases} (S_i^*\otimes I_{\mathcal{D}_*})f \oplus \Delta_{\Theta_T} g & \text{if } i = j, \\ (S_i^*\otimes I_{\mathcal{D}_*})f \oplus 0 & \text{if } i \neq j, \end{cases}$$

$i, j = 1, \dots, n$, and $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$.

Moreover, T is a pure row contraction if and only if Θ_T is an inner multi-analytic operator. In this case the model reduces to

$$\mathbf{H} = (F^2(H_n)\otimes\mathcal{D}_*) \ominus \Theta_T(F^2(H_n)\otimes\mathcal{D}), \qquad \mathbf{T}_i^* f = (S_i^*\otimes I_{\mathcal{D}_*})f, \qquad f \in \mathbf{H}.$$

Any contractive multi-analytic operator $\Theta: F^2(H_n)\otimes\mathcal{E} \to F^2(H_n)\otimes\mathcal{E}_*$ ($\mathcal{E}, \mathcal{E}_*$ are Hilbert spaces) generates a c.n.c. row contraction $\mathbf{T} := [\mathbf{T}_1, \dots, \mathbf{T}_n]$. More precisely, we proved in [8] the following result.

Theorem 2.2. *Let $\Theta: F^2(H_n)\otimes\mathcal{E} \to F^2(H_n)\otimes\mathcal{E}_*$ be a contractive multi-analytic operator and set $\Delta_\Theta := (I - \Theta^*\Theta)^{1/2}$. Then the row contraction $\mathbf{T} := [\mathbf{T}_1, \dots, \mathbf{T}_n]$ defined on the Hilbert space*

$$\mathbf{H} := [(F^2(H_n)\otimes\mathcal{E}_*) \oplus \overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal{E})}] \ominus \{\Theta g \oplus \Delta_\Theta g : \quad g \in F^2(H_n)\otimes\mathcal{E}\}$$

by

$$\mathbf{T}_i^*(f \oplus \Delta_\Theta g) := (S_i^*\otimes I_{\mathcal{E}_*})f \oplus C_i^*(\Delta_\Theta g), \qquad i = 1, \dots, n,$$

where each operator C_i *is defined by*

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i \otimes I_{\mathcal{E}})g, \quad g \in F^2(H_n) \otimes \mathcal{E},$$

and $S_1, \ldots, S_n$ *are the left creation operators on* $F^2(H_n)$*, is completely non-coisometric.*

If Θ *is purely contractive and*

$$\overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})} = \overline{\Delta_\Theta((F^2(H_n) \otimes \mathcal{E}) \ominus \mathcal{E})},$$

then Θ *coincides with the characteristic function of the row contraction* $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$*. In this case, considering* $\mathbf{H}$ *as a subspace of*

$$\mathbf{K} := (F^2(H_n) \otimes \mathcal{E}_*) \oplus \overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})},$$

we have that the sequence of operators $\mathbf{V} := [\mathbf{V}_1, \ldots, \mathbf{V}_n]$ *defined on* $\mathbf{K}$ *by*

$$\mathbf{V}_i := (S_i \otimes I_{\mathcal{E}_*}) \oplus C_i, \qquad i = 1, \ldots, n,$$

is the minimal isometric dilation of $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ *.*

3. Factorizations of characteristic functions and joint invariant subspaces

In this section, we establish the existence of a "one-to-one" correspondence between the joint invariant subspaces under $T_1, \ldots, T_n$, and the regular factorizations of the characteristic function Θ_T associated with a completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$. In particular, we prove that there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$, if and only if there is a non-trivial regular factorization of Θ_T. Using the model theory for c.n.c. row contractions, we provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function.

Let $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ be a contractive multi-analytic operator and assume that it has the factorization

$$\Theta = \Theta_2 \Theta_1,$$

where $\Theta_1 : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{F}$ and $\Theta_2 : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{E}_*$ are contractive multi-analytic operators. Define the operator

$$X_\Theta : \overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})} \to \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}$$

by setting

$$X_\Theta(\Delta_\Theta f) := \Delta_2 \Theta_1 f \oplus \Delta_1 f, \qquad f \in F^2(H_n) \otimes \mathcal{E}, \tag{3.1}$$

where $\Delta_\Theta := (I - \Theta^*\Theta)^{1/2}$ and $\Delta_j := (I - \Theta_j^*\Theta_j)^{1/2}$, $j = 1, 2$. Notice that X_Θ is an isometry. Indeed, since

$$\begin{aligned} I - \Theta^*\Theta &= I - \Theta_1^*\Theta_2^*\Theta_2\Theta_1 \\ &= \Theta_1^*(I - \Theta_2^*\Theta_2)\Theta_1 + (I - \Theta_1^*\Theta_1), \end{aligned}$$

we have

$$\begin{aligned} \|\Delta_2\Theta_1 f \oplus \Delta_1 f\|^2 &= \|\Delta_2\Theta_1 f\|^2 + \|\Delta_1 f\|^2 \\ &= \langle \Theta_1^*(I - \Theta_2^*\Theta_2)\Theta_1 f + I - \Theta_1^*\Theta_1 f, f\rangle \\ &(I - \Theta^*\Theta)f, f\rangle = \|\Delta_\Theta f\|^2. \end{aligned}$$

As in the classical case (see [25]), we say that the factorization $\Theta = \Theta_2\Theta_1$ is *regular* if X_Θ is a unitary operator, i.e.,

$$\left\{\Delta_2\Theta_1 f \oplus \Delta_1 f : \ f \in F^2(H_n) \otimes \mathcal{E}\right\}^- = \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}.$$

Now let us prove the following technical result which will be very useful in what follows.

Lemma 3.1. *Let $\Theta : F^2(H_n)\otimes\mathcal{E}\to F^2(H_n)\otimes\mathcal{E}_*$ be a contractive multi-analytic operator and let $C := [C_1,\ldots,C_n]$ be the row isometry defined on $\overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal{E})}$ by setting*

$$C_i\Delta_\Theta f := \Delta_\Theta(S_i\otimes I_\mathcal{E})f,\quad f\in F^2(H_n)\otimes\mathcal{E},$$

for each $i=1,\ldots,n$, where $\Delta_\Theta := (I-\Theta^*\Theta)^{1/2}$. Then C is a Cuntz row isometry, i.e., $C_1C_1^*+\cdots+C_nC_n^*=I$, if and only if

$$\overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal{E})}=\overline{\Delta_\Theta((F^2(H_n)\otimes\mathcal{E})\ominus\mathcal{E})}. \tag{3.2}$$

Assume that Θ has the factorization

$$\Theta=\Theta_2\Theta_1,$$

where $\Theta_1 : F^2(H_n)\otimes\mathcal{E}\to F^2(H_n)\otimes\mathcal{F}$ and $\Theta_2 : F^2(H_n)\otimes\mathcal{F}\to F^2(H_n)\otimes\mathcal{E}_*$ are contractive multi-analytic operators and let $E:=[E_1,\ldots,E_n]$ and $F:=[F_1,\ldots,F_n]$ be the corresponding row isometries defined on $\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}$ and $\overline{\Delta_2(F^2(H_n)\otimes\mathcal{F})}$, respectively. Then

$$X_\Theta C_i=\begin{pmatrix}F_i & 0\\ 0 & E_i\end{pmatrix}X_\Theta,\quad i=1,\ldots,n, \tag{3.3}$$

where the operator X_Θ is defined by relation (3.1). Moreover, if the factorization $\Theta=\Theta_2\Theta_1$ is regular, then C is a Cuntz row isometry if and only if E and F are Cuntz row isometries.

Proof. First, notice that since Θ is a multi-analytic operator, i.e.,

$$\Theta(S_i\otimes I_\mathcal{E})=(S_i\otimes I_{\mathcal{E}_*})\Theta,\quad i=1,\ldots,n,$$

we have

$$\begin{aligned}\langle C_i\Delta_\Theta f, C_j\Delta_\Theta g\rangle &= \left\langle (S_j^*\otimes I_\mathcal{E})(I-\Theta^*\Theta)(S_i\otimes I_\mathcal{E})f, g\right\rangle\\ &= \langle \delta_{ij}(I-\Theta^*\Theta)f,g\rangle=\delta_{ij}\langle\Delta_\Theta f,\Delta_\Theta g\rangle\end{aligned}$$

for any $f,g\in F^2(H_n)\otimes\mathcal{E}$ and $i,j=1,\ldots,n$. This shows that the operators $C_1,\ldots,C_n$ are isometries with orthogonal spaces. Due to the definition of C_i, it is clear that $C_1C_1^*+\cdots+C_nC_n^*=I$ if and only if the range of the operator $[C_1,\ldots,C_n]$ coincides with $\overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal{E})}$, which is equivalent to (3.2).

On the other hand, for each $i=1,\ldots,n$, and $f\in F^2(H_n)\otimes E$, we have

$$\begin{aligned}X_\Theta C_i(\Delta_\Theta f) &= X_\Theta\Delta_\Theta(S_i\otimes I_\mathcal{E})f\\ &=\Delta_2\Theta_1(S_i\otimes I_\mathcal{E})f\oplus\Delta_1(S_i\otimes I_cE)f\\ &=\Delta_2(S_i\otimes I_\mathcal{F})\Theta_1 f\oplus\Delta_1(S_i\otimes I_\mathcal{E})f\\ &=F_i\Delta_2\Theta_1 f\oplus E_i\Delta_1 f\\ &=\begin{pmatrix}F_i & 0\\ 0 & E_i\end{pmatrix}(\Delta_2\Theta_1 f\oplus\Delta_1 f)\\ &=\begin{pmatrix}F_i & 0\\ 0 & E_i\end{pmatrix}X_\Theta\Delta_\Theta f,\end{aligned}$$

which proves relation (3.3). If the factorization $\Theta=\Theta_2\Theta_1$ is regular, then X_Θ is a unitary operator. Consequently, we have

$$X_\Theta\left(\sum_{i=1}^n C_iC_i^*\right)X_\Theta^*=\begin{pmatrix}\sum_{i=1}^n F_iF_i^* & 0\\ 0 & \sum_{i=1}^n E_iE_i^*\end{pmatrix},$$

which implies that $C := [C_1, \ldots, C_n]$ is a Cuntz row isometry if and only if $E := [E_1, \ldots, E_n]$ and $F := [F_1, \ldots, F_n]$ are Cuntz row isometries. This completes the proof. $\square$

The main result of this section is the following.

Theorem 3.2. *Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a completely non-coisometric row contraction and let $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ be a contractive multi-analytic operator which coincides with the characteristic function of T. If $\mathcal{H}_1 \subset \mathcal{H}$ is a joint invariant subspace under the operators $T_1, \ldots, T_n$, then there exists a regular factorization $\Theta = \Theta_2\Theta_1$, where $\Theta_1 : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{F}$ and $\Theta_2 : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{E}_*$ are contractive multi-analytic operators such that $T := [T_1, \ldots, T_n]$ is unitarily equivalent to a row contraction $\mathbb{T} := [\mathbb{T}_1, \ldots, \mathbb{T}_n]$ defined on the Hilbert space*

$$\begin{aligned}\mathbb{H} := [(F^2(H_n) \otimes \mathcal{E}_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}]\\ \ominus \{\Theta_2\Theta_1 f \oplus \Delta_2\Theta_1 f \oplus \Delta_1 f : \ f \in F^2(H_n) \otimes \mathcal{E}\},\end{aligned}$$

by setting

$$\mathbb{T}_i^*(f \oplus \varphi \oplus \psi) := (S_i^* \otimes I_{\mathcal{E}_*})f \oplus F_i^*\varphi \oplus E_i^*\psi, \qquad f \oplus \varphi \oplus \psi \in \mathbb{H},$$

for any $i = 1, \ldots, n$, where the operators F_i and E_i are defined in Lemma 3.1 and $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$. Moreover, the subspaces corresponding to $\mathcal{H}_1$ and $\mathcal{H}_2 := \mathcal{H} \ominus \mathcal{H}_1$ are

$$\begin{aligned}\mathbb{H}_1 := \{\Theta_2 f \oplus \Delta_2 f \oplus g : \ f \in F^2(H_n) \otimes \mathcal{F}, \ g \in \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}\}\\ \ominus \left\{\Theta_2\Theta_1 f \oplus \Delta_2\Theta_1 f \oplus \Delta_1 f : \ f \in F^2(H_n) \otimes \mathcal{E}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{H}_2 := [(F^2(H_n) \otimes \mathcal{E}_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \{0\}]\\ \ominus \left\{\Theta_2 f \oplus \Delta_2 f \oplus \{0\}) : \ f \in F^2(H_n) \otimes \mathcal{F}\right\},\end{aligned}$$

respectively. Conversely, every regular factorization $\Theta = \Theta_2\Theta_1$ generates via the above formulas the subspaces $\mathbb{H}_1$ and $\mathbb{H}_2$ with the following properties:

(i) *$\mathbb{H}_1$ is invariant under each operator $\mathbb{T}_i$, $i = 1, \ldots, n$;*
(ii) *$\mathbb{H}_2 = \mathbb{H} \ominus \mathbb{H}_1$.*

Under the above identification, $\mathbb{H}_1$ corresponds to a subspace $\mathcal{H}_1 \subset \mathcal{H}$ which is invariant under each operator T_i, $i = 1, \ldots, n$.

Proof. **Part I.** Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction and let $V := [V_1, \ldots, V_n]$, $V_i \in B(\mathcal{K})$, be its the minimal isometric dilation on a Hilbert space $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$. Since $V_1, \ldots, V_n$ are isometries with orthogonal ranges, the noncommutative Wold decomposition [7] provides the orthogonal decomposition

$$\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*), \tag{3.4}$$

where

$$\mathcal{R} := \bigcap_{k=0}^{\infty} \left[\bigoplus_{|\alpha|=k} V_\alpha \mathcal{K}\right] \qquad \text{and} \qquad \mathcal{L}_* := \overline{\left(I_{\mathcal{K}} - \sum_{i=1}^n V_i T_i^*\right)\mathcal{H}}.$$

Moreover, $\mathcal{R}$ is the maximal subspace of $\mathcal{K}$ which is reducing for the operators $V_1, \ldots, V_n$ and the row contraction $[V_1|_{\mathcal{R}}, \ldots, V_n|_{\mathcal{R}}]$ is a Cuntz row isometry.

Let $\mathcal{H}_1 \subset \mathcal{H}$ be an invariant subspace under the operators $T_1, \ldots, T_n$. Since $V_i^*|_{\mathcal{H}} = T_i^*$, $i = 1, \ldots, n$, we deduce that the subspace $\mathcal{H}_2 := \mathcal{H} \ominus \mathcal{H}_1$ is invariant under the operators

$V_1^*, \dots, V_n^*$. Therefore, the subspace $\mathcal{G} := \mathcal{K} \ominus \mathcal{H}_2$ is invariant under $V_1, \dots, V_n$. Applying again the noncommutative Wold decomposition to the row isometry $[V_1|_\mathcal{G}, \dots, V_n|_\mathcal{G}]$, we obtain the orthogonal decomposition

$$\mathcal{G} = \mathcal{R}_1 \oplus M_V(\mathcal{Q}), \tag{3.5}$$

where

$$\mathcal{R}_1 := \bigcap_{k=0}^{\infty} \left[\bigoplus_{|\alpha|=k} V_\alpha \mathcal{G} \right] \quad \text{and} \quad \mathcal{Q} := \mathcal{G} \ominus \left(\bigoplus_{i=1}^{n} V_i \mathcal{G} \right).$$

Since $\mathcal{R}_1$ reduces the operators $V_1, \dots, V_n$ and $[V_1|_{\mathcal{R}_1}, \dots, V_n|_{\mathcal{R}_1}]$ is a Cuntz row isometry, we deduce that $\mathcal{R}_1 \subset \mathcal{R}$. Notice that $\mathcal{R}_2 := \mathcal{R} \ominus \mathcal{R}_1$ is also a reducing subspace for $V_1, \dots, V_n$ and $[V_1|_{\mathcal{R}_2}, \dots, V_n|_{\mathcal{R}_2}]$ is a Cuntz row isometry. Using relations (3.4) and (3.5), we infer that

$$\begin{aligned} \mathcal{H}_2 &= \mathcal{K} \ominus \mathcal{G} \\ &= [\mathcal{R} \oplus M_V(\mathcal{L}_*)] \ominus [\mathcal{R}_1 \oplus M_V(\mathcal{Q})] \\ &= [\mathcal{R}_2 \oplus M_V(\mathcal{L}_*)] \ominus M_V(\mathcal{Q}). \end{aligned}$$

Hence, we deduce that

$$M_V(\mathcal{Q}) \subset \mathcal{R}_2 \oplus M_V(\mathcal{L}_*). \tag{3.6}$$

On the other hand, due to (2.2), we have

$$\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*) = \mathcal{H} \oplus M_V(\mathcal{L}).$$

Hence, we obtain

$$\mathcal{H} = [\mathcal{R} \oplus M_V(\mathcal{L}_*)] \ominus M_V(\mathcal{L}).$$

Since $\mathcal{H}_2 \subset \mathcal{H}$, the above representations of $\mathcal{H}$ and $\mathcal{H}_2$ imply

$$[\mathcal{R}_2 \oplus M_V(\mathcal{L}_*)] \ominus M_V(\mathcal{Q}) \subset [\mathcal{R} \oplus M_V(\mathcal{L}_*)] \ominus M_V(\mathcal{L}).$$

Taking into account that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, we have

$$[\mathcal{R}_2 \oplus M_V(\mathcal{L}_*)] \ominus M_V(\mathcal{Q}) = [\mathcal{R} \oplus M_V(\mathcal{L}_*)] \ominus [\mathcal{R}_1 \oplus M_V(\mathcal{Q})].$$

Consequently, we deduce that

$$M_V(\mathcal{L}) \subset \mathcal{R}_1 \oplus M_V(\mathcal{Q}) \tag{3.7}$$

and

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{H} \ominus \mathcal{H}_2 \\ &= [\mathcal{R}_1 \oplus M_V(\mathcal{Q})] \ominus M_V(\mathcal{L}) \\ &= \mathcal{G} \ominus M_V(\mathcal{L}). \end{aligned}$$

Let $P_{M_V(\mathcal{L}_*)}$, $P_{M_V(\mathcal{Q})}$, $P_\mathcal{R}$, $P_{\mathcal{R}_1}$, and $P_{\mathcal{R}_2}$ be the orthogonal projections onto the corresponding spaces. According to relations (3.6) and (3.7), for any $x \in M_V(\mathcal{Q})$ and $y \in M_V(\mathcal{L})$, we have

$$x = P_{\mathcal{R}_2} x + P_{M_V(\mathcal{L}_*)} x \quad \text{and} \quad y = P_{\mathcal{R}_1} y + P_{M_V(\mathcal{Q})} y. \tag{3.8}$$

In particular, if $x := P_{M_V(\mathcal{Q})} y$ and $y \in M_V(\mathcal{L})$, we deduce that

$$y = P_{\mathcal{R}_1} y + P_{\mathcal{R}_2} P_{M_V(\mathcal{Q})} y + P_{M_V(\mathcal{L}_*)} P_{M_V(\mathcal{Q})} y. \tag{3.9}$$

Hence and taking into account that the subspace $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}$ is orthogonal to $M_V(\mathcal{L}_*)$, we deduce that

$$P_{M_V(\mathcal{L}_*)} y = P_{M_V(\mathcal{L}_*)} P_{M_V(\mathcal{Q})} y \quad \text{and} \quad P_\mathcal{R} y = P_{\mathcal{R}_1} y + P_{\mathcal{R}_2} P_{M_V(\mathcal{Q})} y \tag{3.10}$$

for any $y \in M_V(\mathcal{L})$. Due to relation (3.4), we have

$$P_{\mathcal{R}} f = \left(I - P_{M_V(\mathcal{L}_*)}\right) f, \quad f \in \mathcal{K}. \tag{3.11}$$

On the other hand, relations (3.7) and (3.6) imply

$$P_{\mathcal{R}_1} y = \left(I - P_{M_V(\mathcal{Q})}\right) y, \quad y \in M_V(\mathcal{L}) \tag{3.12}$$

and

$$P_{\mathcal{R}_2} x = \left(I - P_{M_V(\mathcal{L}_*)}\right) x, \quad x \in M_V(\mathcal{Q}). \tag{3.13}$$

Assume now that $[T_1, \ldots, T_n]$ is a c.n.c. row contraction. In this case, we have (see [7])

$$\mathcal{K} = M_V(\mathcal{L}) \bigvee M_V(\mathcal{L}_*) = \mathcal{R} \oplus M_V(\mathcal{L}_*),$$

which implies

$$\overline{P_{\mathcal{R}} M_V(\mathcal{L})} = \overline{\left(I - P_{M_V(\mathcal{L}_*)}\right) M_V(\mathcal{L})} = \mathcal{R}. \tag{3.14}$$

Hence and using the second relation in (3.10), we deduce that

$$\overline{P_{\mathcal{R}_1} M_V(\mathcal{L})} = \mathcal{R}_1 \quad \text{and} \quad \overline{P_{\mathcal{R}_2} P_{M_V(\mathcal{Q})} M_V(\mathcal{L})} = \mathcal{R}_2,$$

and, consequently,

$$\overline{P_{\mathcal{R}_1} M_V(\mathcal{L})} = \mathcal{R}_1 \quad \text{and} \quad \overline{P_{\mathcal{R}_2} M_V(\mathcal{Q})} = \mathcal{R}_2. \tag{3.15}$$

Part II. Consider the following contractions:

$$\begin{aligned}
Q &:= P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{L})} : M_V(\mathcal{L}) \to M_V(\mathcal{L}_*), \\
Q_1 &:= P_{M_V(\mathcal{Q})}|_{M_V(\mathcal{L})} : M_V(\mathcal{L}) \to M_V(\mathcal{Q}), \text{ and} \\
Q_2 &:= P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{Q})} : M_V(\mathcal{Q}) \to M_V(\mathcal{L}_*).
\end{aligned}$$

Since $M_V(\mathcal{L}_*)$, $M_V(\mathcal{L})$, and $M_V(\mathcal{Q})$ are reducing subspaces for the operators $V_1, \ldots, V_n$, we deduce that, for each $i = 1, \ldots, n$,

$$\begin{aligned}
Q \left(V_i|_{M_V(\mathcal{L})}\right) &= \left(V_i|_{M_V(\mathcal{L}_*)}\right) Q, \\
Q_1 \left(V_i|_{M_V(\mathcal{L})}\right) &= \left(V_i|_{M_V(\mathcal{Q})}\right) Q_1, \text{ and} \\
Q_2 \left(V_i|_{M_V(\mathcal{Q})}\right) &= \left(V_i|_{M_V(\mathcal{L}_*)}\right) Q_2.
\end{aligned}$$

Let $\Phi^{\mathcal{L}_*} : M_V(\mathcal{L}_*) \to F^2(H_n) \otimes \mathcal{L}_*$ be the Fourier representation of the subspace $M_V(\mathcal{L}_*)$, i.e.,

$$\Phi^{\mathcal{L}_*}\left(\sum_{\alpha \in \mathbb{F}_n^+} V_\alpha \ell_\alpha\right) := \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes \ell_\alpha,$$

where $\ell_\alpha \in \mathcal{L}_*$ and $\sum_{\alpha \in \mathbb{F}_n^+} \|\ell_\alpha\|^2 < \infty$. Notice that

$$\Phi^{\mathcal{L}_*}\left(V_i|_{M_V(\mathcal{L}_*)}\right) = \left(S_i \otimes I_{\mathcal{L}_*}\right) \Phi^{\mathcal{L}_*}, \quad i = 1, \ldots, n,$$

where $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$. Similarly, we define the Fourier representations of the subspaces $M_V(\mathcal{L})$ and $M_V(\mathcal{Q})$, respectively. Now, due to the above intertwining relations satisfied by Q, Q_1, and Q_2, the operators

$$\begin{aligned}
&\Theta_{\mathcal{L}} : F^2(H_n) \otimes \mathcal{L} \to F^2(H_n) \otimes \mathcal{L}_*, \quad \Theta_{\mathcal{L}} := \Phi^{\mathcal{L}_*} Q (\Phi^{\mathcal{L}})^*, \\
&\Psi_1 : F^2(H_n) \otimes \mathcal{L} \to F^2(H_n) \otimes \mathcal{Q}, \quad \Psi_1 := \Phi^{\mathcal{Q}} Q_1 (\Phi^{\mathcal{L}})^*, \text{ and} \\
&\Psi_2 : F^2(H_n) \otimes \mathcal{Q} \to F^2(H_n) \otimes \mathcal{L}_*, \quad \Psi_2 := \Phi^{\mathcal{L}_*} Q_2 (\Phi^{\mathcal{Q}})^*
\end{aligned} \tag{3.16}$$

are contractive and multi-analytic. Hence and using the first equation in (3.10), we have

$$\begin{aligned}\Theta_{\mathcal{L}} &= \Phi^{\mathcal{L}_*} Q(\Phi^{\mathcal{L}})^* = \Phi^{\mathcal{L}_*}\left(P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{L})}\right)(\Phi^{\mathcal{L}})^* \\ &= \Phi^{\mathcal{L}_*}\left(P_{M_V(\mathcal{L}_*)} P_{M_V(\mathcal{Q})}|_{M_V(\mathcal{L})}\right)(\Phi^{\mathcal{L}})^* \\ &= \left[\Phi^{\mathcal{L}_*}\left(P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{Q})}\right)(\Phi^{\mathcal{Q}})^*\right]\left[\Phi^{\mathcal{Q}}\left(P_{M_V(\mathcal{Q})}|_{M_V(\mathcal{L})}\right)(\Phi^{\mathcal{L}})^*\right] \\ &= \left[\Phi^{\mathcal{L}_*} Q_2 (\Phi^{\mathcal{Q}})^*\right]\left[\Phi^{\mathcal{Q}} Q_1 (\Phi^{\mathcal{L}})^*\right] \\ &= \Psi_2 \Psi_1.\end{aligned}$$

Due to (3.11) and (3.14), there exists a unique unitary operator $\Phi_{\mathcal{R}} : \mathcal{R} \to \overline{\Delta_{\mathcal{L}}(F^2(H_n) \otimes \mathcal{L})}$ such that

$$\Phi_{\mathcal{R}} P_{\mathcal{R}} \psi := \Delta_{\mathcal{L}} \Phi^{\mathcal{L}} \psi, \quad \psi \in M_V(\mathcal{L}), \tag{3.17}$$

where $\Delta_{\mathcal{L}} := (I - \Theta_{\mathcal{L}}^* \Theta_{\mathcal{L}})^{1/2}$. Indeed, we have

$$\begin{aligned}\|(I - P_{M_V(\mathcal{L}_*)})\psi\|^2 &= \|\psi\|^2 - \|P_{M_V(\mathcal{L}_*)}\psi\|^2 \\ &\|\Phi^{\mathcal{L}}\psi\|^2 - \|\Phi^{\mathcal{L}_*} P_{M_V(\mathcal{L}_*)}\psi\|^2 \\ &= \|\Phi^{\mathcal{L}}\psi\|^2 - \|\Theta_{\mathcal{L}} \Phi^{\mathcal{L}}\psi\|^2 \\ &= \|\Delta_{\mathcal{L}} \Phi^{\mathcal{L}}\psi\|^2.\end{aligned}$$

Consequently,

$$\Phi := \Phi^{\mathcal{L}_*} \oplus \Phi_{\mathcal{R}} \tag{3.18}$$

is a unitary operator from the dilation space $\mathcal{K} = M_V(\mathcal{L}_*) \oplus \mathcal{R}$ onto the Hilbert space

$$\widetilde{\mathbf{K}} := (F^2(H_n) \otimes \mathcal{L}_*) \oplus \overline{\Delta_{\mathcal{L}}(F^2(H_n) \otimes \mathcal{L})}.$$

The image of the space $\mathcal{H} = \mathcal{K} \ominus M_V(\mathcal{L})$ under the operator Φ is

$$\Phi\mathcal{H} = \widetilde{\mathbf{H}} := [(F^2(H_n) \otimes \mathcal{L}_*) \oplus \overline{\Delta_{\mathcal{L}}(F^2(H_n) \otimes \mathcal{L})}] \ominus \{\Theta_{\mathcal{L}} f \oplus \Delta_{\mathcal{L}} f \ : \ f \in F^2(H_n) \otimes \mathcal{L}\}.$$

The row contraction $T := [T_1, \dots, T_n]$ is transformed under the unitary operator Φ into the row contraction $\widetilde{\mathbf{T}} := [\widetilde{\mathbf{T}}_1, \dots, \widetilde{\mathbf{T}}_n]$, where

$$\widetilde{\mathbf{T}}_i^*(f \oplus \Delta_{\mathcal{L}} g) := (S_i^* \otimes I_{\mathcal{L}_*}) f \oplus \widetilde{C}_i^*(\Delta_{\mathcal{L}} g), \qquad i = 1, \dots, n,$$

and each operator $\widetilde{C}_i$ is defined by

$$\widetilde{C}_i(\Delta_{\mathcal{L}} g) = \Delta_{\mathcal{L}}(S_i \otimes I_{\mathcal{L}}) g, \quad g \in F^2(H_n) \otimes \mathcal{L}.$$

Notice that, using relations (3.12), (3.13), and (3.15), one can show that there are some unitary operators

$$\Phi_{\mathcal{R}_1} : \mathcal{R}_1 \to \overline{\Delta_{\Psi_1}(F^2(H_n) \otimes \mathcal{L})} \quad \text{and} \quad \Phi_{\mathcal{R}_2} : \mathcal{R}_2 \to \overline{\Delta_{\Psi_2}(F^2(H_n) \otimes \mathcal{Q})}$$

uniquely defined by the relations

$$\begin{aligned}\Phi_{\mathcal{R}_1} P_{\mathcal{R}_1} x &:= \Delta_{\Psi_1} \Phi^{\mathcal{L}} x, \quad x \in M_V(\mathcal{L}), \\ \Phi_{\mathcal{R}_2} P_{\mathcal{R}_2} y &:= \Delta_{\Psi_2} \Phi^{\mathcal{Q}} y, \quad y \in M_V(\mathcal{Q}),\end{aligned} \tag{3.19}$$

where $\Delta_{\Psi_j} := \left(I - \Psi_j^* \Psi_j\right)^{1/2}$ for $j = 1, 2$. Consequently, since $\mathcal{R} = \mathcal{R}_2 \oplus \mathcal{R}_1$ and due to relation (3.17), the operator

$$X_{\mathcal{L}} : \overline{\Delta_{\mathcal{L}}(F^2(H_n) \otimes \mathcal{L})} \to \overline{\Delta_{\Psi_2}(F^2(H_n) \otimes \mathcal{Q})} \oplus \overline{\Delta_{\Psi_1}(F^2(H_n) \otimes \mathcal{L})}$$

defined by

$$X_{\mathcal{L}} := (\Phi_{\mathcal{R}_2} \oplus \Phi_{\mathcal{R}_1}) \Phi_{\mathcal{R}}^* \tag{3.20}$$

is unitary. Due to relations (3.17), (3.10), (3.19), and (3.16), we deduce that

$$\begin{aligned} X_{\mathcal{L}} \Delta_{\mathcal{L}} \Phi^{\mathcal{L}} y &= X_{\mathcal{L}} \Phi_{\mathcal{R}} P_{\mathcal{R}} y = (\Phi_{\mathcal{R}_2} \oplus \Phi_{\mathcal{R}_1}) P_{\mathcal{R}} y \\ &= (\Phi_{\mathcal{R}_2} \oplus \Phi_{\mathcal{R}_1}) \left(P_{\mathcal{R}_2} P_{M_V(\mathcal{Q})} y \oplus P_{\mathcal{R}_1} y\right) \\ &= \Delta_{\Psi_2} \Phi^{\mathcal{Q}} P_{M_V(\mathcal{Q})} y \oplus \Delta_{\Psi_1} \Phi^{\mathcal{L}} y \\ &= \Delta_{\Psi_2} \Psi_1 \Phi^{\mathcal{L}} y \oplus \Delta_{\Psi_1} \Phi^{\mathcal{L}} y \end{aligned}$$

for any $y \in M_V(\mathcal{L})$. Hence, we have

$$X_{\mathcal{L}} \Delta_{\mathcal{L}} f = \Delta_{\Psi_2} \Psi_1 f \oplus \Delta_{\Psi_1} f, \quad f \in F^2(H_n) \otimes \mathcal{L}. \tag{3.21}$$

Since $X_{\mathcal{L}}$ is a unitary operator, we also deduce that

$$\left\{\Delta_{\Psi_2} \Theta_1 f \oplus \Delta_{\Psi_1} f, \quad f \in F^2(H_n) \otimes \mathcal{L}\right\}^- = \overline{\Delta_{\Psi_2}(F^2(H_n) \otimes \mathcal{Q})} \bigoplus \overline{\Delta_{\Psi_1}(F^2(H_n) \otimes \mathcal{L})}.$$

Due to (3.18) and (3.20), we have

$$\Phi = \Phi^{\mathcal{L}_*} \oplus X_{\mathcal{L}}^* (\Phi_{\mathcal{R}_2} \oplus \Phi_{\mathcal{R}_1}).$$

Now, we need to find the images $\widetilde{\mathbf{H}}_1$ and $\widetilde{\mathbf{H}}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, under the unitary operator Φ. To find $\widetilde{\mathbf{H}}_2$, notice first that, due to relation (3.20), we have

$$\Phi_{\mathcal{R}} z = X_{\mathcal{L}}^* (\Phi_{\mathcal{R}_2} \oplus \Phi_{\mathcal{R}_1}) (z \oplus 0) = X_{\mathcal{L}}^* (\Phi_{\mathcal{R}_2} z \oplus 0) \tag{3.22}$$

for any $z \in \mathcal{R}_2$. Hence and using (3.17), we infer that

$$\begin{aligned} \Phi (M_V(\mathcal{L}_*) \oplus \mathcal{R}_2) &= \Phi^{\mathcal{L}_*} M_V(\mathcal{L}_*) \oplus \Phi_{\mathcal{R}} \mathcal{R}_2 \\ &= (F^2(H_n) \otimes \mathcal{L}_*) \bigoplus X_{\mathcal{L}}^* \left(\overline{\Delta_{\Psi_2}(F^2(H_n) \otimes \mathcal{Q})} \oplus \{0\}\right) \end{aligned}$$

and, due to (3.8),

$$\Phi M_V(\mathcal{Q}) = \left\{\Phi^{\mathcal{L}_*} P_{M_V(\mathcal{L}_*)} f \oplus \Phi_{\mathcal{R}} P_{\mathcal{R}_2} f : \ f \in M_V(\mathcal{Q})\right\}.$$

Hence, and using relations (3.16), (3.19), and (3.22), we obtain

$$\Phi M_V(\mathcal{Q}) = \left\{\Psi_2 u \oplus X_{\mathcal{L}}^* (\Delta_{\Psi_2} u \oplus 0) : \ u \in F^2(H_n) \otimes \mathcal{Q}\right\}.$$

Now, using the representation of $\mathcal{H}_2$ from Part I, i.e.,

$$\mathcal{H}_2 = [M_V(\mathcal{L}_*) \oplus \mathcal{R}_2] \ominus M_V(\mathcal{Q}),$$

we obtain

$$\begin{aligned} \widetilde{\mathbf{H}}_2 = [(F^2(H_n) \otimes \mathcal{L}_*) &\bigoplus X_{\mathcal{L}}^* \left(\overline{\Delta_{\Psi_2}(F^2(H_n) \otimes \mathcal{Q})}) \oplus \{0\}\right)] \\ &\ominus \left\{\Psi_2 f \oplus X_{\mathcal{L}}^* (\Delta_{\Psi_2} f \oplus 0) : \ f \in F^2(H_n) \otimes \mathcal{Q}\right\}. \end{aligned}$$

Since $\widetilde{\mathbf{H}}_1 = \widetilde{\mathbf{H}} \ominus \widetilde{\mathbf{H}}_2$, we deduce that

$$\begin{aligned} \widetilde{\mathbf{H}}_1 = \{\Psi_2 f \oplus X_{\mathcal{L}}^* (\Delta_{\Psi_2} f \oplus g) : \ f \in F^2(H_n) \otimes \mathcal{Q}, \ g \in \overline{\Delta_{\Psi_1}(F^2(H_n) \otimes \mathcal{L})}\} \\ \ominus \left\{\Theta_{\mathcal{L}} w \oplus \Delta_{\Theta} w : \ w \in F^2(H_n) \otimes \mathcal{L}\right\}. \end{aligned}$$

According to Section 2, the characteristic function Θ_T of the row contraction T coincides with $\Theta_{\mathcal{L}}$, and therefore with Θ. Via this identification, the regular factorization $\Theta_{\mathcal{L}} = \Psi_2 \Psi_1$ corresponds to a regular factorization $\Theta = \Theta_2 \Theta_1$, where $\Theta_1 : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{F}$ and $\Theta_2 : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{E}_*$ are contractive multi-analytic operators. Now, it is

easy to see that, under the above identification, the subspaces $\widetilde{\mathbf{H}}_1$ and $\widetilde{\mathbf{H}}_2$ correspond to the subspaces

$$\begin{aligned}\mathbf{H}_2 = [(F^2(H_n)\otimes \mathcal{E}_*) \oplus X_\Theta^* \left(\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F}))} \oplus \{0\}\right)] \\ \ominus \left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus 0):\ f\in F^2(H_n)\otimes \mathcal{F}\right\}\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\mathbf{H}_1 = \{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus g):\ f\in F^2(H_n)\otimes \mathcal{F},\ g\in \overline{\Delta_1(F^2(H_n)\otimes \mathcal{E})}\} \\ \ominus \left\{\Theta\varphi \oplus \Delta_\Theta\varphi:\ \varphi\in F^2(H_n)\otimes \mathcal{E}\right\},\end{aligned} \tag{3.24}$$

respectively, where $\Delta_j := (I-\Theta_j^*\Theta_J)^{1/2}$, $j=1,2$.. Moreover, under the same identification, the row contraction $\widetilde{\mathbf{T}}$ is unitarily equivalent to the row contraction $\mathbf{T} := [\mathbf{T}_1,\ldots,\mathbf{T}_n]$ defined on the Hilbert space

$$\mathbf{H} := [(F^2(H_n)\otimes \mathcal{E}_*) \oplus \overline{\Delta_\Theta(F^2(H_n)\otimes \mathcal{E})}] \ominus \{\Theta g\oplus \Delta_\Theta g:\quad g\in F^2(H_n)\otimes \mathcal{E}\},$$

by

$$\mathbf{T}_i^*(f\oplus \Delta_\Theta g) := (S_i^*\otimes I_{\mathcal{E}_*})f \oplus C_i^*(\Delta_\Theta g),\qquad i=1,\ldots,n,$$

where each operator C_i is defined by

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i\otimes I_{\mathcal{E}})g,\quad g\in F^2(H_n)\otimes \mathcal{E},$$

and $S_1,\ldots,S_n$ are the left creation operators on $F^2(H_n)$.

Since the factorization $\Theta=\Theta_2\Theta_1$ is regular, X_Θ is a unitary operator which identifies the subspace $\overline{\Delta_\Theta(F^2(H_n)\otimes \mathcal{E})}$ with $\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}\oplus\overline{\Delta_1(F^2(H_n)\otimes \mathcal{E})}$ and the operator C_i with $\begin{pmatrix} F_i & 0\\ 0 & E_i\end{pmatrix}$, for each $i=1,\ldots,n$. Under this identification the Hilbert spaces $\mathbf{H}$, $\mathbf{H_1}$, and $\mathbf{H_2}$ are identified with $\mathbb{H},\mathbb{H}_1$, and $\mathbb{H}_2$, respectively, and the row contraction $\mathbf{T}$ is unitarily equivalent to the row contraction $\mathbb{T}$.

Part III. We prove the converse of the theorem. Due to the above identification, it is enough to assume that the factorization $\Theta=\Theta_2\Theta_1$ is regular and the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ are defined as above by relations (3.24) and (3.23), respectively. Since X_Θ is a unitary operator and using the definition (3.1), we have

$$\begin{aligned}\mathbf{G}_2 :&= \left\{\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g):\ f\in F^2(H_n)\otimes \mathcal{F},\ g\in \overline{\Delta_1(F^2(H_n)\otimes \mathcal{E})}\right\}\\ &\supset \left\{\Theta_2\Theta_1\varphi \oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus \Delta_1\varphi):\ \varphi\in F^2(H_n)\otimes \mathcal{E}\right\}\\ &= \left\{\Theta\varphi + \Delta_\Theta\varphi:\ \varphi\in F^2(H_n)\otimes \mathcal{E}\right\}\end{aligned}$$

Hence, we obtain

$$\mathbf{H}_1 = \mathbf{G}_2 \ominus \left\{\Theta\varphi+\Delta_\Theta\varphi:\ \varphi\in F^2(H_n)\otimes\mathcal{E}\right\}.$$

On the other hand, we have

$$\begin{aligned}&\left[(F^2(H_n)\ \otimes\mathcal{E}_*) \oplus \overline{\Delta_\Theta(F^2(H_n)\otimes \mathcal{E})}\right] \ominus \mathbf{G}_2\\ &\qquad= \left[(F^2(H_n)\otimes \mathcal{E}_*)\oplus X_\Theta^*\left(\overline{\Delta_2(F^2(H_n)\otimes\mathcal{F})}\oplus\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\right)\right]\ominus \mathbf{G}_2\\ &\qquad= \left[(F^2(H_n)\otimes \mathcal{E}_*)\oplus X_\Theta^*\left(\overline{\Delta_2(F^2(H_n)\otimes\mathcal{F})}\oplus\{0\}\right)\right]\\ &\qquad\qquad\qquad \ominus\left\{\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus\{0\}) : f\in F^2(H_n)\otimes\mathcal{F}\right\}.\end{aligned}$$

Consequently,

$$\mathbf{H}_2 = \left[(F^2(H_n)\otimes\mathcal{E}_*)\oplus\overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal{E})}\right]\ominus \mathbf{G}_2.$$

Hence, and taking into account the definition of $\mathbf{H}_1$, we deduce that $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$.

It remains to prove that the subspace $\mathbf{H}_2$ is invariant under the operators $\mathbf{T}_1^*, \ldots, \mathbf{T}_n^*$. If $f \in F^2(H_n) \otimes \mathcal{E}_*$ and $g \in \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})}$, then the vector $x := f \oplus X_\Theta^*(g \oplus 0)$ is in $\mathbf{H}_2$ if and only if

$$\Theta_2^* f + \Delta_2 g = 0. \tag{3.25}$$

Indeed, using relation (3.23), one can prove that the condition

$$\langle f \oplus X_\Theta^*(g \oplus 0), \Theta_2 \varphi \oplus X_\Theta^*(\Delta_2 \varphi \oplus 0)\rangle = 0 \quad \text{for any} \quad \varphi \in F^2(H_n) \otimes \mathcal{F}$$

is quivalent to (3.25). Since

$$\begin{aligned}\mathbf{T}_i^* x &= \mathbf{T}_i^*(f \oplus X_\Theta^*(g \oplus 0)) \\ &= (S_i^* \otimes I_{\mathcal{E}_*}) f \oplus C_i^* X_\Theta^*(g \oplus 0)\end{aligned}$$

for each $i = 1, \ldots, n$, to prove that $\mathbf{T}_i^* x \in \mathbf{H}_2$, it is enough to show that

$$\langle (S_i^* \otimes I_{\mathcal{E}_*}) f \oplus C_i^*(X_\Theta^*(g \oplus 0)), \Theta_2 \varphi \oplus X_\Theta^*(\Delta_2 \varphi \oplus 0)\rangle = 0$$

for any $\varphi \in F^2(H_n) \otimes \mathcal{F}$. Since Θ is a multi-analytic operator, the latter condition is equivalent to

$$(S_i^* \otimes I_{\mathcal{F}})\Theta_2^* f + \Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) = 0, \tag{3.26}$$

where P_1 is the orthogonal projection of the direct sum $\overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}$ onto $\overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})}$. Using Lemma 3.1 and the definition of the operators C_i, E_i, and F_i, we deduce that

$$\begin{aligned}\Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) &= \Delta_2 P_1 X_\Theta X_\Theta^* \begin{pmatrix} F_i^* & 0 \\ 0 & E_i^* \end{pmatrix} (g \oplus 0) \\ &= \Delta_2 F_i^* g \\ &= (S_i^* \otimes I_{\mathcal{F}}) \Delta_2 g.\end{aligned}$$

Hence, and using relation (3.25), we have

$$(S_i^* \otimes I_{\mathcal{F}})\Theta_2^* f + \Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) = (S_i^* \otimes I_{\mathcal{F}})(\Theta_2^* f + \Delta_2 g) = 0,$$

which proves relation (3.26). This shows that $\mathbf{T}_i^* \mathbf{H}_2 \subset \mathbf{H}_2$ for any $i = 1, \ldots, n$. Consequently, the subspace $\mathbf{H}_1 = \mathbf{H} \ominus \mathbf{H}_2$ is invariant under the operators $\mathbf{T}_1, \ldots, \mathbf{T}_n$. This completes the proof of the theorem. □

Now we can reformulate Theorem 3.2 in terms of the functional model of a c.n.c. row contraction provided by Theorem 2.2. This version will be useful later on.

Theorem 3.3. *Let* $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ *be a purely contractive multi-analytic operator such that*

$$\overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})} = \overline{\Delta_\Theta[(F^2(H_n) \otimes \mathcal{E}) \ominus \mathcal{E}]}.$$

and let $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ *be defined on the Hilbert space*

$$\mathbf{H} := [(F^2(H_n) \otimes \mathcal{E}_*) \oplus \overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})}] \ominus \{\Theta g \oplus \Delta_\Theta g : \quad g \in F^2(H_n) \otimes \mathcal{E}\},$$

by

$$\mathbf{T}_i^*(f \oplus \Delta_\Theta g) := (S_i^* \otimes I_{\mathcal{E}_*}) f \oplus C_i^*(\Delta_\Theta g), \qquad i = 1, \ldots, n,$$

where each operator C_i *is defined by*

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i \otimes I_{\mathcal{E}}) g, \qquad g \in F^2(H_n) \otimes \mathcal{E},$$

and $S_1, \ldots, S_n$ *are the left creation operators on* $F^2(H_n)$.

If $\mathbf{H}_1 \subseteq \mathbf{H}$ is an invariant subspace under each operator $\mathbf{T}_i$, $i = 1, \ldots, n$, then there is a regular factorization

$$\Theta = \Theta_2 \Theta_1,$$

where $\Theta_1 : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{F}$ and $\Theta_2 : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{E}_*$ are contractive multi-analytic operators such that, if X_Θ is the operator defined by (3.1)*, then the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2 := \mathbf{H} \ominus \mathbf{H}_1$ have the representations:*

$$\begin{aligned}\mathbf{H}_1 = \{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) :\ f \in F^2(H_n) \otimes \mathcal{F},\ g \in \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}\} \\ \ominus \left\{\Theta\varphi \oplus \Delta_\Theta \varphi :\ \varphi \in F^2(H_n) \otimes \mathcal{E}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}_2 = [(F^2(H_n) \otimes \mathcal{E}_*) \oplus X_\Theta^* \left(\overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})}) \oplus \{0\}\right)] \\ \ominus \left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus 0) :\ f \in F^2(H_n) \otimes \mathcal{F}\right\}.\end{aligned}$$

Conversely, every regular factorization $\Theta = \Theta_2\Theta_1$ generates via the above formulas the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ with the following properties:

(i) *$\mathbf{H}_1$ is an invariant subspace under each operator $\mathbf{T}_i$, $i = 1, \ldots, n$;*
(ii) *$\mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1$.*

In what follows we need the following factorization result for contractive multi-analytic operators [18].

Lemma 3.4. *Let $\Theta \in R_n^\infty \bar{\otimes} B(\mathcal{E}, \mathcal{E}')$ be a contractive multi-analytic operator. Then Θ admits a unique decomposition $\Theta = \Phi \oplus \Lambda$ with the following properties:*

(i) *$\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{E}_0, \mathcal{E}_0')$ is purely contractive, i.e., $\|P_{\mathcal{E}_0'} \Psi h\| < \|h\|$ for any $h \in \mathcal{E}_0$, $h \neq 0$;*
(ii) *$\Lambda = I \otimes U \in R_n^\infty \bar{\otimes} B(\mathcal{E}_u, \mathcal{E}_u')$, where $U \in B(\mathcal{E}_u, \mathcal{E}_u')$ is a unitary operator;*
(iii) *$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_u$ and $\mathcal{E}' = \mathcal{E}_0' \oplus \mathcal{E}_u'$.*

Moreover, the purely contractive part of an outer or inner multi-analytic operator is also outer or inner, respectively.

The next result is an addition to Theorem 2.2.

Proposition 3.5. *Let $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ be a contractive multi-analytic operator such that*

$$\overline{\Delta_\Theta(F^2(H_n) \otimes \mathcal{E})} = \overline{\Delta_\Theta[(F^2(H_n) \otimes \mathcal{E}) \ominus \mathcal{E}]}.$$

and let $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ be the functional model associated with Θ, as in Theorem 2.2.

(i) *The characteristic function of $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ coincides with the purely contractive part of Θ.*
(ii) *The space $\mathbf{H}$ defined in Theorem 2.2 is different from $\{0\}$ if and only if there is no unitary operator $U \in B(\mathcal{E}, \mathcal{E}_*)$ such that $\Theta = I \otimes U$.*

Proof. According to Lemma 3.4, the multi-analytic operator Θ admits the decomposition $\Theta = \Phi \oplus \Lambda$ with $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{E}_0, \mathcal{E}_{*0})$ purely contractive and $\Lambda = I \otimes U \in R_n^\infty \bar{\otimes} B(\mathcal{E}_u, \mathcal{E}_{*u})$, where $U \in B(\mathcal{E}_u, \mathcal{E}_{*u})$ is a unitary operator, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_u$, and $\mathcal{E}_* = \mathcal{E}_{*0} \oplus \mathcal{E}_{*u}$. Notice that

$$F^2(H_n) \otimes \mathcal{E}_* = (F^2(H_n) \otimes \mathcal{E}_{*u}) \oplus (F^2(H_n) \otimes \mathcal{E}_{*0}) \text{ and}$$
$$F^2(H_n) \otimes \mathcal{E} = (F^2(H_n) \otimes \mathcal{E}_u) \oplus (F^2(H_n) \otimes \mathcal{E}_0).$$

On the other hand, we have

$$\left\{\Theta g \oplus \Delta_\Theta g : \ g \in F^2(H_n) \otimes \mathcal{E}\right\} = (F^2(H_n) \otimes \mathcal{E}_{*u}) \oplus \left\{\Phi\varphi \oplus \Delta_\Phi \varphi : \ \varphi \in F^2(H_n) \otimes \mathcal{E}_0\right\}.$$

Now, using the definition of the Hilbert space $\mathbf{H}$, one can identify $\mathbf{H}$ with

$$\mathbf{H}_0 := \left[(F^2(H_n) \otimes \mathcal{E}_{*0}) \oplus \overline{\Delta_\Phi(F^2(H_n) \otimes \mathcal{E}_0)}\right] \ominus \left\{\Phi\varphi \oplus \Delta_\Phi \varphi : \ \varphi \in F^2(H_n) \otimes \mathcal{E}_0\right\}.$$

Due to this identification, the row contraction $\mathbf{T} := [\mathbf{T}_1, \dots, \mathbf{T}_n]$ is unitarily equivalent to $\mathbf{T}^0 := [\mathbf{T}_1^0, \dots, \mathbf{T}_n^0]$, which is defined on $\mathbf{H}_0$ in the same manner as $\mathbf{T}$ is defined on $\mathbf{H}$. Since $\Delta_\Theta = \Delta_\Phi \oplus 0$, it is easy to see that

$$\overline{\Delta_\Phi(F^2(H_n) \otimes \mathcal{E})} = \overline{\Delta_\Phi[(F^2(H_n) \otimes \mathcal{E}) \ominus \mathcal{E}]}.$$

According to the second part of Theorem 2.2 the characteristic function of $\mathbf{T}^0$ coincides with the multi-analytic operator Φ which coincides with the characteristic function of $\mathbf{T}$.

We prove now part (ii). If $\Theta = I \otimes U$ for some unitary operator $U \in B(\mathcal{E}, \mathcal{E}_*)$, then $\Delta_\Theta = 0$ and

$$\mathbf{H} = [F^2(H_n) \otimes \mathcal{E}_*] \ominus \Theta(F^2(H_n) \otimes \mathcal{E}) = \{0\}.$$

If Θ is not a unitary multi-analytic operator, then, according to Lemma 3.4, it has a non-trivial purely contractive part. By part (i), Theorem 2.1, and Theorem 2.2, we deduce that

$$\dim \mathcal{D}_* = \dim \mathcal{E}_{*0}, \quad \dim \mathcal{D} = \dim \mathcal{E}_0,$$

where $\mathcal{E}$ and $\mathcal{E}_{*0}$ are not both equal to $\{0\}$. Since $\mathcal{D}_* \subset \mathcal{H}$ and $\mathcal{D} \subset \mathcal{H}^{(n)}$, we deduce that $\mathcal{H} \neq \{0\}$. This completes the proof. □

The following result is an important addition to Theorem 3.3 (and hence also to Theorem 3.2).

Theorem 3.6. *Under the conditions of Theorem 3.3, let $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$ be the decomposition corresponding to the regular factorization $\Theta = \Theta_2\Theta_1$, and let*

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{A}_i & * \\ 0 & \mathbf{B}_i \end{pmatrix}, \quad i = 1, \dots, n,$$

be the corresponding triangulation of $\mathbf{T} := [\mathbf{T}_1, \dots, \mathbf{T}_n]$. Then the characteristic functions of the row contractions $\mathbf{A} := [\mathbf{A}_1, \dots, \mathbf{A}_n]$ and $\mathbf{B} := [\mathbf{B}_1, \dots, \mathbf{B}_n]$ coincide with the purely contractive parts of the multi-analytic operators Θ_1 and Θ_2, respectively.

Moreover, the invariant subspace $\mathbf{H}_1$ under the operators $\mathbf{T}_1, \dots, \mathbf{T}_n$ is non-trivial if and only if the regular factorization $\Theta = \Theta_2\Theta_1$ is non-trivial, i.e., each factor is not a unitary constant.

Proof. Define the operator U from the Hilbert space

$$(F^2(H_n) \otimes \mathcal{E}_*) \oplus X_\Theta^* \left(\overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})} \oplus \{0\}\right)$$

to

$$(F^2(H_n) \otimes \mathcal{E}_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes \mathcal{F})}$$

by setting

$$U(f \oplus X^*(g \oplus 0)) := f \oplus g,$$

for any $f\in F^2(H_n)\otimes \mathcal{E}_*$ and $g\in \overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}$. Since X_Θ is unitary, so is U. Using the definition of $\mathbf{H}_2$ (see relation (3.23)), we deduce that $U\mathbf{H}_2=\widehat{\mathcal{H}}_2$, where

$$\begin{aligned}\widehat{\mathcal{H}}_2 := \Big[(F^2(H_n)\otimes \mathcal{E}_*) \oplus \overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}\Big]&\\ \ominus \left\{\Theta_2\varphi\oplus \Delta_2\varphi:\ \varphi\in F^2(H_n)\otimes \mathcal{F}\right\}.&\end{aligned} \tag{3.27}$$

Set $\Gamma_i^*:=U\mathbf{B}_i^*U^*$, $i=1,\dots,n$, and denote by P_1 the orthogonal projection of the direct sum $\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}\oplus \overline{\Delta_1(F^2(H_n)\otimes \mathcal{E})}$ onto $\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}$. Using Lemma 3.1, we deduce that

$$\begin{aligned}P_1X_\Theta C_i^*X_\Theta^*(g\oplus 0)&=P_1\begin{pmatrix}F_i^* & 0\\ 0 & E_i^*\end{pmatrix}\begin{pmatrix} g\\ 0\end{pmatrix}\\ &=F_i^*g\end{aligned}$$

for any $g\in \overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}$ and $i=1,\dots,n$. Hence and using the definitions for the row contraction $[\mathbf{T}_1,\dots,\mathbf{T}_n]$ and the unitary operator U, we have

$$\begin{aligned}\Gamma_i^*(f\oplus g)&=U\mathbf{T}_i^*(f\oplus X_\Theta^*(g\oplus 0))\\ &=U\left[(S_i^*\otimes I_{\mathcal{E}_*})f\oplus C_i^*X_\Theta^*(g\oplus 0)\right]\\ &=(S_i^*\otimes I_{\mathcal{E}_*})f\oplus P_1X_\Theta C_i^*X_\Theta^*(g\oplus 0)\\ &=(S_i^*\otimes I_{\mathcal{E}_*})f\oplus F_i^*g\end{aligned}$$

for any $f\in F^2(H_n)\otimes \mathcal{E}_*$ and $g\in \overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}$ such that $f\oplus g\in \mathcal{H}_2$, and $i=1,\dots,n$.

Since

$$\overline{\Delta_\Theta(F^2(H_n)\otimes \mathcal{E})}=\overline{\Delta_\Theta(F^2(H_n)\otimes \mathcal{E})\ominus \mathcal{E}},$$

one can use again Lemma 3.1 to deduce that

$$\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})}=\overline{\Delta_2(F^2(H_n)\otimes \mathcal{F})\ominus \mathcal{F}}.$$

Now, due to Proposition 3.5, we infer that the characteristic function of the row contraction $[\Gamma_1,\dots,\Gamma_n]$, $\Gamma_i\in B(\widehat{\mathcal{H}}_2)$, (and hence also $[\mathbf{B}_1,\dots,\mathbf{B}_n]$) coincides with the purely contractive part of the multi-analytic operator Θ_2.

Taking into account the definition of the subspace $\mathbf{H}_1$ (see relation (3.24)) and the fact that $\Theta=\Theta_2\Theta_1$, one can see that, for each $f\in F^2(H_n)\otimes \mathcal{F}$ and $g\in \overline{\Delta_1(F^2(H_n)\otimes \mathcal{E})}$, the vector $\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g)$ is in $\mathbf{H}_1$ if and only if

$$\langle \Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g), \Theta_2\Theta_1\varphi\oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus\Delta_1\varphi)\rangle=0$$

for any $\varphi\in F^2(H_n)\otimes \mathcal{E}$. The latter equation is equivalent to

$$\Theta_1^*\Theta_2^*\Theta_2 f+\Theta_1^*\Delta_2^2 f+\Delta_1 g=0.$$

Since $\Delta_2^2=I-\Theta_2^*\Theta_2$, the above equation is equivalent to

$$\Theta_1^* f+\Delta_1 g=0. \tag{3.28}$$

If $x:=\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g)\in \mathbf{H}_1$, then we have

$$\mathbf{T}_i^*x=(S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f\oplus C_i^*X_\Theta^*(\Delta_2 f\oplus g)$$

for each $i=1,\dots,n$. Since Θ_2 is a multi-analytic operator and

$$f=\sum_{j=1}^n (S_jS_j^*\otimes I_{\mathcal{F}})f+f(0),$$

where $f(0) := P_{1\otimes \mathcal{F}} f$, we deduce that

$$\begin{aligned}\mathbf{T}_i^* x &= [\Theta_2(S_i^*\otimes I_{\mathcal{F}})f + (S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0)] \oplus C_i^* X_\Theta^*(\Delta_2 f\oplus g)\\ &= u+v,\end{aligned}$$

where

$$u := \Theta_2(S_i^*\otimes I_{\mathcal{F}})f \oplus [X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus E_i^* g)]$$

and

$$v := (S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0) \oplus [C_i^* X_\Theta^*(\Delta_2 f\oplus g) - X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus E_i^* g)]\,.$$

Now notice that $u\in \mathbf{H}_1$. Indeed, using the above characterization of the elements of $\mathbf{H}_1$, it is enough to show that

$$\Theta_1^*(S_i^*\otimes I_{\mathcal{F}})f + \Delta_1 E_i^* g = 0, \quad i=1,\dots,n. \tag{3.29}$$

Using relation (3.28) and the definition of E_i, we have

$$\begin{aligned}\Theta_1^*(S_i^*\otimes I_{\mathcal{F}})f + \Delta_1 E_i^* g &= (S_i^*\otimes I_{\mathcal{E}})(\Theta_1^* f + \Delta_1 g)\\ &= 0,\end{aligned}$$

which proves (3.29) and therefore $u\in\mathbf{H}_1$.

Now we prove that $v\in\mathbf{H}_2$. First, notice that due to Lemma 3.1, we have

$$C_i^* X_\Theta^*(0\oplus g) = X_\Theta^*(0\oplus E_i^* g), \quad g\in \overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})},$$

and therefore

$$v = (S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0) \oplus [C_i^* X_\Theta^*(\Delta_2 f\oplus 0) - X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus 0)]\,. \tag{3.30}$$

Using again Lemma 3.1, and the definition of F_i, we infer that

$$\begin{aligned}C_i^* X_\Theta^*(\Delta_2 f\oplus 0) &= C_i^* X_\Theta^*\left(\Delta_2\left(\sum_{j=1}^n S_j S_j^*\otimes I_{\mathcal{F}}\right) f(0)\oplus 0\right) + C_i^* X_\Theta^*(\Delta_2 f(0)\oplus 0)\\ &= X_\Theta^*\left(F_i^*\Delta_2\left(\sum_{j=1}^n S_j S_j^*\otimes I_{\mathcal{F}}\right) f\oplus 0\right) + C_i^* X_\Theta^*(\Delta_2 f(0)\oplus 0)\\ &= X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus 0) + C_i^* X_\Theta^*(\Delta_2 f(0)\oplus 0)\\ &= X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus 0) + X_\Theta^*(F_i^*\Delta_2 f(0)\oplus 0).\end{aligned}$$

Consequently, relation (3.30) implies

$$v = (S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0)\oplus X_\Theta^*(F_i^*\Delta_2 f(0)\oplus 0).$$

Due to the definition of the subspace $\mathbf{H}_2$, to prove that $v\in\mathbf{H}_2$, it is enough to show that

$$\Theta_2^*(S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0) + \Delta_2 F_i^*\Delta_2 f(0) = 0$$

for each $i=1,\dots,n$. Since

$$\Delta_2 F_i^* = (S_i^*\otimes I_{\mathcal{F}})\Delta_2, \quad i=1,\dots,n,$$

and Θ_2 is multi-analytic, we have

$$\begin{aligned}\Theta_2^*(S_i^*\otimes I_{\mathcal{E}_*})\Theta_2 f(0) + \Delta_2 F_i^*\Delta_2 f(0) &= (S_i^*\otimes I_{\mathcal{F}})(\Theta_2^*\Theta_2 + \Delta_2^2) f(0)\\ &= (S_i^*\otimes I_{\mathcal{F}})f(0) = 0.\end{aligned}$$

Hence, $v\in\mathbf{H}_2$. Now, using the fact that $\mathbf{T}_i^* x = u+v$ and the definitions for u and v, we deduce that the operator $\mathbf{A}_i^* := P_{\mathbf{H}_1}\mathbf{T}_i^*|_{\mathbf{H}_1}$ satisfies the equation

$$\mathbf{A}_i^*\left(\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g)\right) = \Theta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus [X_\Theta^*(\Delta_2(S_i^*\otimes I_{\mathcal{F}})f\oplus E_i^* g)] \tag{3.31}$$

for any $\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) \in \mathbf{H}_1$ and $i = 1, \ldots, n$.

Now, define the operator Ω from

$$\left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) : \ f \in F^2(H_n) \otimes \mathcal{F}, g \in \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}\right\}$$

to the direct sum $(F^2(H_n) \otimes \mathcal{F}) \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}$ by setting

$$\Omega(\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g)) := f \oplus g. \tag{3.32}$$

Since

$$\begin{aligned}\|\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g)\|^2 &= \|\Theta_2 f\|^2 + \|X_\Theta^*(\Delta_2 f \oplus g)\|^2 \\ &= \langle \Theta_2^* \Theta_2 f, f\rangle + \|\Delta_2 f\|^2 + \|g\|^2 \\ &= \|f \oplus g\|^2,\end{aligned}$$

it is clear that Ω is a unitary operator. Notice also that

$$\begin{aligned}\Omega(\Theta\varphi \oplus \Delta_\Theta \varphi) &= \Omega(\Theta_2\Theta_1\varphi \oplus X_\Theta^*(\Delta_2\Theta_1\varphi \oplus \Delta_1\varphi)) \\ &= \Theta_1\varphi \oplus \Delta_1\varphi\end{aligned}$$

for any $\varphi \in F^2(H_n) \otimes \mathcal{E}$. Consequently, $\Omega \mathbf{H}_1 = \widehat{\mathcal{H}}_1$, where

$$\begin{aligned}\widehat{\mathcal{H}}_1 := {} & [(F^2(H_n) \otimes \mathcal{F}) \oplus \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})}] \\ & \ominus \left\{\Theta_1\varphi \oplus \Delta_1\varphi : \ \varphi \in F^2(H_n) \otimes \mathcal{E}\right\}.\end{aligned} \tag{3.33}$$

Setting $\Lambda_i := \Omega \mathbf{A_i} \Omega^*$, relation (3.31) implies

$$\Lambda_i^*(f \oplus g) = (S_i^* \otimes I_\mathcal{F}) f \oplus E_i^* g, \quad f \oplus g \in \mathcal{H}_1,$$

for any $i = 1, \ldots, n$. Once again, Lemma 3.1 implies

$$\overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})} = \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E}) \ominus \mathcal{E}}.$$

Now, using Proposition 3.5, we infer that the characteristic function of the row contraction $[\Lambda_1, \ldots, \Lambda_n]$, $\Lambda_i \in B(\widehat{\mathcal{H}}_1)$, (and hence also $[\mathbf{A}_1, \ldots, \mathbf{A}_n]$) coincides with the purely contractive part of the multi-analytic operator Θ_1. Due to the relations (3.27), (3.33), and Proposition 3.5, the subspaces $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_2$ (and hence also $\mathbf{H}_1$ and $\mathbf{H}_2$) are different from $\{0\}$ if and only if both multi-analytic operators Θ_1 and Θ_2 are not unitary constant, i.e., the factorization $\Theta = \Theta_1\Theta_2$ is non-trivial. This completes the proof. □

Now, combining Theorem 3.2 and Theorem 3.6, we can deduce the following result.

Theorem 3.7. *Let $T := [T_1, \ldots, T_n]$ be a completely non-coisometric row contraction on a separable Hilbert space $\mathcal{H}$. Then, there is a non-trivial invariant subspace under each operator $T_1, \ldots, T_n$ if and only if the characteristic function Θ_T has a non-trivial regular factorization.*

Concerning the uniqueness in Theorem 3.3 (and also Theorem 3.2), we can prove the following result, which shows the extent to which a joint invariant subspace determines the corresponding regular factorization of the characteristic function.

Theorem 3.8. *Under the conditions of Theorem 3.3, let*

$$\Theta = \Theta_2\Theta_1 \quad \text{and} \quad \Theta = \Theta_2'\Theta_1'$$

be two regular factorizations of the purely contractive multi-analytic operator Θ, and let $\mathcal{E}, \mathcal{F}, \mathcal{E}_*$, and $\mathcal{E}, \mathcal{F}', \mathcal{E}_*$ be the corresponding Hilbert spaces. Let $\mathbf{H}_1 \subset \mathbf{H}$ and $\mathbf{H}_1' \subset \mathbf{H}$ be the

invariant subspaces under each operator $\mathbf{T}_i$, $i = 1, \ldots, n$, corresponding to the above factorizations. If $\mathbf{H}_1 \subset \mathbf{H}_1'$, then there is a multi-analytic operator $\Psi : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{F}'$ such that

$$\Theta_1' = \Psi \Theta_1.$$

Moreover, if $\mathbf{H}_1 = \mathbf{H}_1'$, then

$$\Theta_1' = (I \otimes \Psi_0)\Theta_1$$

for some unitary operator $\Psi_0 \in B(\mathcal{F}, \mathcal{F}')$ and, consequently, the multi-analytic operators Θ_1 and Θ_1' coincide.

Proof. We associate with the factorization $\Theta = \Theta_2\Theta_1$ the subspace

$$\mathcal{M} := \left\{ \Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) : \ f \in F^2(H_n) \otimes \mathcal{F}, g \in \overline{\Delta_1(F^2(H_n) \otimes \mathcal{E})} \right\}.$$

Similarly, we define the subspace $\mathcal{M}'$ associated with the factorization $\Theta = \Theta_2'\Theta_1'$. Since $\mathbf{H}_1 \subseteq \mathbf{H}_1'$, relation (3.24) and its analogue for $\mathbf{H}_1'$ imply $\mathcal{M} \subseteq \mathcal{M}'$. Consequently, for each $f \in F^2(H_n) \otimes \mathcal{F}$, there exist $f' \in F^2(H_n) \otimes \mathcal{F}'$ and $g' \in \overline{\Delta_1'(F^2(H_n) \otimes \mathcal{E})}$ such that

$$\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus 0) = \Theta_2' f' \oplus X_\Theta'^*(\Delta_2 f' \oplus g'). \tag{3.34}$$

Hence and using the definition of the unitary operators X_Θ and X_Θ', we have

$$\begin{aligned}
\|f\|^2 &= \|\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g)\|^2 \\
&= \|\Theta_2' f' \oplus X_\Theta'^*(\Delta_2 f' \oplus g')\|^2 \\
&= \|f'\|^2 + \|g'\|^2.
\end{aligned}$$

Therefore, it makes sense to define the contractions $Q : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{F}'$ and $R : F^2(H_n) \otimes \mathcal{F} \to \overline{\Delta_1'(F^2(H_n) \otimes \mathcal{E})}$ by setting $Qf := f'$ and $Rf := g'$, respectively. Now, we show that Q is a multi-analytic operator, i.e.,

$$Q(S_i \otimes I_\mathcal{F}) = (S_i \otimes I_{\mathcal{F}'})Q, \quad i = 1, \ldots, n.$$

Let $f_1, \ldots, f_n$ be arbitrary elements in $F^2(H_n) \otimes \mathcal{E}$. Taking into account the definitions for C_i and X_Θ, and the fact that

$$(S_j^* \otimes I_\mathcal{F})\Delta_2^2(S_i \otimes I_\mathcal{F}) = \delta_{ij}\Delta_2^2, \quad i, j = 1, \ldots, n,$$

we deduce that

$$\begin{aligned}
\left\langle C_i X_\Theta^*(\Delta_2 f \oplus 0), \Delta_\Theta \left(\sum_{j=1}^n (S_j \otimes I_\mathcal{E}) f_j \right) \right\rangle &= \langle \Delta_2 f \oplus 0), X_\Theta \Delta_\Theta f_i \rangle \\
&= \langle \Delta_2 f \oplus 0), \Delta_2 \Theta_1 f_i \oplus \Delta_1 f_i \rangle \\
&= \left\langle \Delta_2^2 f, \Theta_1 f_i \right\rangle
\end{aligned}$$

and

$$\left\langle X_\Theta^*(\Delta_2(S_i\otimes I_{\mathcal F})f\oplus 0), \Delta_\Theta\left(\sum_{j=1}^n (S_j\otimes I_{\mathcal E})f_j\right)\right\rangle$$

$$\begin{aligned}
&= \left\langle \Delta_2(S_i\otimes I_{\mathcal F})f\oplus 0, \Delta_2\Theta_1\left(\sum_{j=1}^n (S_j\otimes I_{\mathcal E})f_j\right)\oplus \Delta_1\left(\sum_{j=1}^n (S_j\otimes I_{\mathcal E})f_j\right)\right\rangle\\
&= \left\langle \Delta_2(S_i\otimes I_{\mathcal F})f, \Delta_2\Theta_1\left(\sum_{j=1}^n (S_j\otimes I_{\mathcal E})f_j\right)\right\rangle\\
&= \sum_{j=1}^n \left\langle (S_j^*\otimes I_{\mathcal F})\Delta_2^2(S_i\otimes I_{\mathcal F})f, \Theta_1 f_j\right\rangle\\
&= \left\langle \Delta_2^2 f, \Theta_1 f_i\right\rangle.
\end{aligned}$$

Hence, and taking into account that

$$\overline{\Delta_\Theta(F^2(H_n)\otimes\mathcal E)} = \overline{\Delta_\Theta[(F^2(H_n)\otimes\mathcal E)\ominus\mathcal E]},$$

we deduce that

$$C_iX_\Theta^*(\Delta_2 f\oplus 0) = X_\Theta^*(\Delta_2(S_i\otimes I_{\mathcal F})f\oplus 0) \quad \text{for any} \quad f\in F^2(H_n)\otimes\mathcal F. \tag{3.35}$$

Similar calculations show that

$$C_iX_\Theta^*(0\oplus\Delta_1\varphi) = X_\Theta^*(0\oplus\Delta_1(S_i\otimes I_{\mathcal E})\varphi) \tag{3.36}$$

for any $\varphi\in F^2(H_n)\otimes\mathcal E$ and $i=1,\dots,n$. Moreover, similar relations to (3.35) and (3.36) hold with X'_Θ, Δ'_1, and Δ'_2 instead of X_Θ, Δ_1, and Δ_2, respectively. Since

$$C_iX_\Theta'^*(0\oplus\Delta'_1\varphi) = X_\Theta'^*(0\oplus\Delta'_1(S_i\otimes I_{\mathcal E})\varphi) \tag{3.37}$$

for any $\varphi\in F^2(H_n)\otimes\mathcal E$ and $i=1,\dots,n$, by taking appropriate limits, we deduce that

$$C_iX_\Theta'^*(\{0\}\oplus\overline{\Delta'_1(F^2(H_n)\otimes\mathcal E)})\subseteq X_\Theta'^*(\{0\}\oplus\overline{\Delta'_1(F^2(H_n)\otimes\mathcal E)}).$$

Consequently, for each $g'\in\overline{\Delta'_1(F^2(H_n)\otimes\mathcal E)}$ there exists $g''\in\overline{\Delta'_1(F^2(H_n)\otimes\mathcal E)}$ such that

$$C_iX_\Theta'^*(0\oplus g') = X_\Theta'^*(0\oplus g''). \tag{3.38}$$

Now, notice that using relations (3.35), (3.34), (3.37), and (3.38), we obtain

$$\begin{aligned}
\Theta_2(S_i\otimes I_{\mathcal F})f\oplus X_\Theta^*(\Delta_2(S_i\otimes I_{\mathcal F})f\oplus 0) &= (S_i\otimes I_{\mathcal E_*}\oplus C_i)(\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus 0))\\
&= (S_i\otimes I_{\mathcal E_*}\oplus C_i)(\Theta'_2 f'\oplus X_\Theta'^*(\Delta'_2 f'\oplus g'))\\
&= \Theta'_2(S_i\otimes I_{\mathcal F'})f'\oplus X_\Theta'^*(\Delta'_2(S_i\otimes I_{\mathcal F'})f'\oplus g'')
\end{aligned}$$

for any $f\in F^2(H_n)\otimes\mathcal F$. Hence and using the definition of Q, we deduce that

$$Q(S_i\otimes I_{\mathcal F})f = (S_i\otimes I_{\mathcal F'})f' = (S_i\otimes I_{\mathcal F'})Qf, \quad f\in F^2(H_n)\otimes\mathcal F,$$

which proves that Q is a multi-analytic operator.

Since $\mathcal M\subset\mathcal M'$, we have

$$\bigcap_{k=0}^\infty \bigoplus_{|\alpha|=k}[(S_\alpha\otimes I_{\mathcal E_*})\oplus C_\alpha]\mathcal M \subseteq \bigcap_{k=0}^\infty \bigoplus_{|\alpha|=k}[(S_\alpha\otimes I_{\mathcal E_*})\oplus C_\alpha]\mathcal M'. \tag{3.39}$$

Using Lemma 3.1, the definition (3.32) of the unitary operator Ω, and relations (3.35), (3.36), one can prove that

$$[(S_i\otimes I_{\mathcal{E}_*})\oplus C_i]\Omega^*=\Omega^*[(S_i\otimes I_{\mathcal{F}})\oplus E_i].$$

Indeed, we have

$$\begin{aligned}[(S_i\otimes I_{\mathcal{E}_*})\oplus C_i]\Omega^*(f\oplus \Delta_1\varphi)&=\Theta_2(S_i\otimes I_{\mathcal{F}})f\oplus C_iX_\Theta^*(\Delta_2 f\oplus \Delta_1\varphi)\\&=\Theta_2(S_i\otimes I_{\mathcal{F}})f\oplus X_\Theta^*(\Delta_2(S_i\otimes I_{\mathcal{F}})f\oplus \Delta_1(S_i\otimes I_{\mathcal{E}})\varphi)\\&=\Omega^*[(S_i\otimes I_{\mathcal{F}})f\oplus \Delta_1(S_i\otimes I_{\mathcal{E}})\varphi)]\\&=\Omega^*[(S_i\otimes I_{\mathcal{F}})\oplus E_i](f\oplus\Delta_1\varphi)\end{aligned}$$

for any $f\in F^2(H_n)\otimes\mathcal{F}$ and $\varphi\in F^2(H_n)\otimes\mathcal{E}$.

Now, due to the fact that $[S_1\otimes I_{\mathcal{F}},\dots,S_n\otimes I_{\mathcal{F}}]$ is a multi-shift and $[E_1,\dots,E_n]$ is a Cuntz row isometry, the noncommutative Wold decomposition implies

$$\begin{aligned}&\bigcap_{k=0}^\infty\bigoplus_{|\alpha|=k}[(S_\alpha\otimes I_{\mathcal{E}_*})\oplus C_\alpha]\mathcal{M}\\&\quad=\Omega^*\left\{\bigcap_{k=0}^\infty\left[\bigoplus_{|\alpha|=k}(S_\alpha\otimes I_{\mathcal{F}})(F^2(H_n)\otimes\mathcal{F})\right]\oplus\bigcap_{k=0}^\infty\left[\bigoplus_{|\alpha|=k}E_\alpha\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\right]\right\}\\&\quad=\Omega^*\left(\{0\}\oplus\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\right)\\&\quad=\left\{0\oplus X_\Theta^*(0\oplus g):\ g\in\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\right\}.\end{aligned}$$

A similar relation can be obtain for the set on the right side of the inclusion (3.39). Hence and using relation (3.39), we obtain

$$\left\{0\oplus X_\Theta^*(0\oplus g):\ g\in\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\right\}\subseteq\left\{0\oplus X_\Theta'^*(0\oplus g'):\ g'\in\overline{\Delta_1'(F^2(H_n)\otimes\mathcal{E})}\right\}.$$

Consequently, for each $g\in\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}$ there exists $g'\in\overline{\Delta_1'(F^2(H_n)\otimes\mathcal{E})}$ such that

$$X_\Theta^*(0\oplus g)=X_\Theta'^*(0\oplus g'). \tag{3.40}$$

Since X_Θ and X_Θ' are unitary operators, we can define the isometry

$$V:\overline{\Delta_1(F^2(H_n)\otimes\mathcal{E})}\to\overline{\Delta_1'(F^2(H_n)\otimes\mathcal{E})}$$

by setting $Vg:=g'$. For each $\varphi\in F^2(H_n)\otimes\mathcal{E}$, we have

$$\Theta\varphi\oplus\Delta_\Theta\varphi=\Theta_2'\Theta_1'\varphi\oplus X_\Theta'^*(\Delta_2'\Theta_1'\varphi\oplus\Delta_1'\varphi). \tag{3.41}$$

On the other hand, using the operators Q,R,V and relation (3.34), we deduce that

$$\begin{aligned}\Theta\varphi\oplus\Delta_\Theta\varphi&=\Theta_2\Theta_1\varphi\oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus\Delta_1\varphi)\\&=[\Theta_2\Theta_1\varphi\oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus 0]+[0\oplus X_\Theta^*(0\oplus\Delta_1\varphi)]\\&=\left[\Theta_2'Q\Theta_1\varphi\oplus X_\Theta'^*(\Delta_2'Q\Theta_1\varphi\oplus R\Theta_1\varphi)\right]+\left[0\oplus X_\Theta'^*(0\oplus V\Delta_1\varphi)\right]\\&=\Theta_2'Q\Theta_1\varphi\oplus X_\Theta'^*(\Delta_2'Q\Theta_1\varphi\oplus y),\end{aligned}$$

where $y:=R\Theta_1\varphi+V\Delta_1\varphi$ is in $\overline{\Delta_1'(F^2(H_n)\otimes\mathcal{E})}$. Using the latter relation and (3.41), we obtain

$$\Theta_2'\Theta_1'\varphi=\Theta_2'Q\Theta_1\varphi\quad\text{ and }\quad\Delta_2'\Theta_1'\varphi=\Delta_2'Q\Theta_1\varphi.$$

Since the mapping $\Theta_2'f'\oplus\Delta_2'f'\mapsto f'$ is isometric, we deduce that

$$\Theta_1'\varphi=Q\Theta_1\varphi,\quad \varphi\in F^2(H_n)\otimes\mathcal{E}, \tag{3.42}$$

which proves the first part of the theorem.

Now assume that $\mathbf{H}_1 = \mathbf{H}_1'$. A closer look at the above proof reveals that $Q(F^2(H_n)\otimes\mathcal{F}) = F^2(H_n)\otimes\mathcal{F}'$ and V is a unitary operator. Taking into account relation (3.40) and (3.34), we obtain

$$\begin{aligned}\Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus 0) &= \left[\Theta_2' f' \oplus X_\Theta'^*(\Delta_2' f'\oplus 0)\right] + \left[0\oplus X_\Theta'^*(0\oplus g')\right]\\ &= \left[\Theta_2' f' \oplus X_\Theta'^*(\Delta_2' f'\oplus 0)\right] + \left[0\oplus X_\Theta^*(0\oplus V^* g')\right].\end{aligned}$$

Hence, we get

$$\Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus (-V^*g')) = \Theta_2' f' \oplus X_\Theta'^*(\Delta_2' f'\oplus 0).$$

Taking the norms, we have

$$\|f\|^2 + \|g'\|^2 = \|f'\|^2.$$

Combining this with $\|f\|^2 = \|f'\|^2 + \|g'\|^2$, we obtain $\|f\| = \|f'\|$, which shows that Q is a unitary multi-analytic operator. Due to [13], this implies $Q = I\otimes\Psi_0$, for some unitary operator $\Psi_0\in B(\mathcal{F},\mathcal{F}')$. Using relation (3.42), we complete the proof. □

4. Triangulations for row contractions and joint invariant subspaces

In this section, we prove the existence of a unique triangulation of type

$$\begin{pmatrix} C_{\cdot 0} & 0\\ * & C_{\cdot 1}\end{pmatrix} \tag{4.1}$$

for any row contraction $T := [T_1,\dots,T_n]$, and prove the existence of joint invariant subspaces for certain classes of row contractions.

We need a few definitions. Let $T := [T_1,\dots,T_n]$, $T_i\in B(\mathcal{H})$, be a row contraction. We say that T is of class $C_{\cdot 0}$ (or pure row contraction) if

$$\lim_{k\to\infty}\sum_{|\alpha|=k}\|T_\alpha^* h\|^2 = 0 \quad \text{for any} \quad h\in\mathcal{H},$$

and of class $C_{\cdot 1}$ if

$$\lim_{k\to\infty}\sum_{|\alpha|=k}\|T_\alpha^* h\|^2 \neq 0 \quad \text{for any} \quad h\in\mathcal{H},\ h\neq 0.$$

We say that a row contraction $T := [T_1,\dots,T_n]$, $T_i\in B(\mathcal{H})$, has a triangulation of type (4.1) if there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0\oplus\mathcal{H}_1$ with respect to which

$$T_i = \begin{pmatrix} A_i & 0\\ * & B_i\end{pmatrix}, \quad i = 1,\dots,n,$$

and the entries have the following properties:

(i) $T_i^*\mathcal{H}_0\subset\mathcal{H}_0$ for any $i = 1,\dots,n$;
(ii) $A := [A_1,\dots,A_n]$ is of class $C_{\cdot 0}$;
(iii) $B := [B_1,\dots,B_n]$ is of class $C_{\cdot 1}$.

The type of the entry denoted by $*$ is not specified.

Theorem 4.1. *Every row contraction $T := [T_1,\dots,T_n]$, $T_i\in B(\mathcal{H})$, has a triangulation of type*

$$\begin{pmatrix} C_{\cdot 0} & 0\\ * & C_{\cdot 1}\end{pmatrix}$$

Moreover, this triangulation is uniquely determined.

Proof. First, notice that the subspace

$$\mathcal{H}_0 := \left\{ h \in \mathcal{H} : \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = 0 \right\}$$

is invariant under each operator T_i^*, $i = 1, \ldots, n$. The decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_1 := \mathcal{H} \ominus \mathcal{H}_0$, yields the triangulation

$$T_i^* = \begin{pmatrix} A_i^* & * \\ 0 & B_i^* \end{pmatrix}, \quad i = 1, \ldots, n,$$

where $A_i^* := T_i^*|_{\mathcal{H}_0}$ and $B_i^* := P_{\mathcal{H}_1} T_i^*|_{\mathcal{H}_1}$ for each $i = 1, \ldots, n$. Since

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \|A_\alpha^* h\|^2 = \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = 0, \quad h \in \mathcal{H}_0,$$

the row contraction $A := [A_1, \ldots, A_n]$ is of class $C_{\cdot 0}$. Now, we need to show that

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \|B_\alpha^* h\|^2 \neq 0 \quad \text{for all} \quad h \in \mathcal{H}_1, h \neq 0.$$

Lt $V := [V_1, \ldots, V_n]$, $V_i \in B(\mathcal{K})$, be the minimal isometric dilation of the row contraction $T := [T_1, \ldots, T_n]$ (see Section 2). For every $m = 1, \ldots,$ the isometries V_α, $|\alpha| = m$, have orthogonal ranges. Therefore, we have

$$\begin{aligned} \left\| \sum_{|\alpha|=m} V_\alpha \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{\mathcal{H}_0} T_\alpha^* h \right\|^2 &= \sum_{|\alpha|=m} \left\| \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{\mathcal{H}_0} T_\alpha^* h \right\|^2 \\ &= \sum_{|\alpha|=m} \sum_{|\beta|=k} \left\| T_\beta^* P_{\mathcal{H}_0} T_\alpha^* h \right\|^2 \end{aligned}$$

for any $h \in \mathcal{H}$. Since $P_{\mathcal{H}_0} T_\alpha^* h \in \mathcal{H}_0$, we have

$$\lim_{k\to\infty} \sum_{|\beta|=k} \left\| T_\beta^* P_{\mathcal{H}_0} T_\alpha^* h \right\|^2 = 0. \tag{4.2}$$

According to [7], we have

$$P_{\mathcal{R}} h = \lim_{k\to\infty} \sum_{|\alpha|=k} V_\alpha T_\alpha^* h, \quad \text{for any} \quad h \in \mathcal{H}, \tag{4.3}$$

where $P_{\mathcal{R}}$ is the orthogonal projection of the minimal isometric dilation space $\mathcal{K}$ on the subspace $\mathcal{R}$ in the Wold decomposition $\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*)$. Now, using relations (4.2) and (4.3), we obtain

$$\begin{aligned} P_{\mathcal{R}} h &= \lim_{k\to\infty} \sum_{|\alpha|=m} \sum_{|\beta|=k} V_\alpha V_\beta T_\beta^* T_\alpha^* h \\ &= \lim_{k\to\infty} \sum_{|\alpha|=m} V_\alpha \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{\mathcal{H}_0} T_\alpha^* h + \lim_{k\to\infty} \sum_{|\alpha|=m} V_\alpha \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{\mathcal{H}_1} T_\alpha^* h \\ &= \sum_{|\alpha|=m} V_\alpha P_{\mathcal{R}} P_{\mathcal{H}_1} T_\alpha^* h. \end{aligned}$$

Hence, we deduce that

$$\|P_{\mathcal{R}}h\|^2 = \left\|\sum_{|\alpha|=m} V_\alpha P_{\mathcal{R}} P_{\mathcal{H}_1} T_\alpha^* h\right\|^2 = \sum_{|\alpha|=m} \|P_{\mathcal{R}} P_{\mathcal{H}_1} T_\alpha^* h\|^2$$
$$\leq \sum_{|\alpha|=m} \|P_{\mathcal{H}_1} T_\alpha^* h\|^2 = \sum_{|\alpha|=m} \|B_\alpha^* h\|^2$$

for any $h \in \mathcal{H}$. Let $h \in \mathcal{H}_1$, $h \neq 0$, and assume that $\lim_{m\to\infty} \sum_{|\alpha|=m} \|B_\alpha^* h\|^2 = 0$. The above relation shows that $P_{\mathcal{R}}h = 0$ and, due to (4.3), we deduce that $h \in \mathcal{H}_0$, which is a contradiction.

Now, we prove the uniqueness. Assume that there is another decomposition $\mathcal{H} = \mathcal{H}_0' \oplus \mathcal{H}_1'$ which yields the triangulation

$$T_i = \begin{pmatrix} A_i' & 0 \\ * & B_i' \end{pmatrix}, \quad i = 1, \ldots, n,$$

of type $\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix}$, where $A_i'^* := T_i^*|_{\mathcal{H}_0'}$ and $B_i'^* := P_{\mathcal{H}_1'} T_i^*|_{\mathcal{H}_1'}$ for each $i = 1, \ldots, n$. To prove uniqueness, it is enough to show that $\mathcal{H}_0 = \mathcal{H}_0'$. Notice that if $h \in \mathcal{H}_0'$, then, due to the fact that the row contraction $[A_1', \ldots, A_n']$ is of class $C_{\cdot 0}$, we have

$$\lim_{m\to\infty} \sum_{|\alpha|=m} \|T_\alpha^* h\|^2 = \lim_{m\to\infty} \sum_{|\alpha|=m} \|{A_\alpha'}^* h\|^2 = 0.$$

Hence, $h \in \mathcal{H}_0$, which proves that $\mathcal{H}_0' \subseteq \mathcal{H}_0$. Assume now that $h \in \mathcal{H}_0 \ominus \mathcal{H}_0'$. Since $h \in \mathcal{H}_1'$, we have

$$\lim_{m\to\infty} \sum_{|\alpha|=m} \|{B_\alpha'}^* h\|^2 = \lim_{m\to\infty} \sum_{|\alpha|=m} \|P_{\mathcal{H}_1'} {T_\alpha}^* h\|^2 \leq \lim_{m\to\infty} \sum_{|\alpha|=m} \|T_\alpha^* h\|^2 = 0.$$

Consequently, since the row contraction $[B_1', \ldots, B_n']$ is of class $C_{\cdot 1}$, we must have $h = 0$. Hence, we deduce that $\mathcal{H}_0 \ominus \mathcal{H}_0' = \{0\}$, which shows that $\mathcal{H}_0' = \mathcal{H}_0$. This completes the proof. □

Corollary 4.2. *If $T := [T_1, \ldots, T_n]$ is a row contraction such $T \notin C_{\cdot 0}$ and $T \notin C_{\cdot 1}$, then there is a non-trivial joint invariant subspace under $T_1, \ldots, T_n$.*

According to Section 2, any row contraction admits a triangulation of type

$$\begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}$$

where C_c (resp. C_{cnc}) denotes the class of coisometric (resp. c.n.c.) row contractions. Notice that $C_c \subset C_{\cdot 1}$. Combining this result with the triangulation of Theorem 4.1, we obtain another triangulation for row contractions, that is,

$$\begin{pmatrix} C_{\cdot 0} & 0 & 0 \\ * & C_c & 0 \\ * & * & C_{cnc} \cap C_{\cdot 1} \end{pmatrix}.$$

Corollary 4.3. *If $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, is a row contraction such*

$$T_1 T_1^* + \cdots + T_n T_n^* \neq I$$

and there is a non-zero vector $h \in \mathcal{H}$ such that $\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2$ for any $k = 1, 2, \ldots$, then there is a non-trivial subspace under the operators $T_1, \ldots, T_n$.

We recall from [17] that if

$$T_1T_1^* + \cdots + T_nT_n^* = I,$$

then a subspace $\mathcal{M}$ is invariant under $T_1, \ldots, T_n$ if and only if

$$T_1P_{\mathcal{M}}T_1^* + \cdots + T_nP_{\mathcal{M}}T_n^* \le P_{\mathcal{M}},$$

where $P_{\mathcal{M}}$ is the orthogonal projection on $\mathcal{M}$. We also mention that the case when $T \in C_{\cdot 0}$ is treated in the next corollary, and the case $T \in C_{\cdot 1}$ is considered in the next section (see Theorem 5.5).

The proof of the following result on regular factorizations of multi-analytic operators is straightforward from the definition, so we leave it to the reader.

Lemma 4.4. *Let $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ be a contractive multi-analytic operator and assume that it has the factorization*

$$\Theta = \Theta_2\Theta_1,$$

where $\Theta_1 : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{F}$ and $\Theta_2 : F^2(H_n) \otimes \mathcal{F} \to F^2(H_n) \otimes \mathcal{E}_*$ are contractive multi-analytic operators.

(i) *If Θ_2 is inner, then the factorization $\Theta = \Theta_2\Theta_1$ is regular.*
(ii) *If Θ is inner, then the factorization $\Theta = \Theta_2\Theta_1$ is regular if and only if Θ_1 and Θ_2 are inner multi-analytic operators.*
(iii) *If $\operatorname{rank} \Delta_\Theta < \infty$, then*

$$\operatorname{rank} \Delta_\Theta = \operatorname{rank} \Delta_{\Theta_2} + \operatorname{rank} \Delta_{\Theta_1}$$

if and only if the factorization $\Theta = \Theta_2\Theta_1$ is regular.

Now we consider the case when T is a pure row contraction.

Corollary 4.5. *If $T := [T_1, \ldots, T_n]$ is a row contraction of class $C_{\cdot 0}$, then the non-trivial joint invariant subspaces under $T_1, \ldots, T_n$ are parametrized by the non-trivial inner factorizations of the characteristic function Θ_T of T (i.e., $\Theta_T = \Theta_2\Theta_1$ with Θ_1 and Θ_2 inner multi-analytic operators). Moreover, the subspaces $\mathbb{H}_1$ and $\mathbb{H}_2$ in Theorem 3.2 become*

$$\mathbb{H}_1 = \{\Theta_2 f : \ f \in F^2(H_n) \otimes \mathcal{F}\} \ominus \{\Theta_T f : \ f \in F^2(H_n) \otimes \mathcal{D}\} \text{ and}$$
$$\mathbb{H}_2 = \{F^2(H_n) \otimes \mathcal{D}_*\} \ominus \{\Theta_2 f : \ f \in F^2(H_n) \otimes \mathcal{F}\},$$

where $\mathcal{D}$ and $\mathcal{D}_*$ are the defect spaces of T.

Proof. According to Theorem 2.1, the characteristic function Θ_T is an inner multi-analytic operator. By Lemma 4.4, any factorization $\Theta_T = \Theta_2\Theta_1$ is regular if and only if Θ_1 and Θ_2 are inner operators. Applying now Theorem 3.2, in our particular case, the result follows. □

We should remark that Corollary 4.5 can also be proved directly using Theorem 2.1 and the Beurling type characterization (see [8]) of the joint invariant subspaces under the operators $S_1 \otimes I_{\mathcal{G}}, \ldots, S_n \otimes I_{\mathcal{G}}$.

We recall [9] that any multi-analytic operator admits an essentially unique inner-outer factorization.

Theorem 4.6. *Let $T := [T_1, \ldots, T_n]$ be a completely non-coisometric row contraction. The inner-outer factorization of the characteristic function Θ_T induces (cf. Theorem 3.6) the triangulation of type*

$$\begin{pmatrix} C_{\cdot 0} & 0 \\ * & C_{\cdot 1} \end{pmatrix}$$

for the row contraction T.

In particular, if the inner-outer factorization of the characteristic function is non-trivial, then there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$.

Proof. Suppose that the multi-analytic operator $\Theta : F^2(H_n) \otimes \mathcal{E} \to F^2(H_n) \otimes \mathcal{E}_*$ coincides with the characteristic function of the c.n.c. row contraction $T := [T_1, \ldots, T_n]$. Let $\Theta = \Theta_i \Theta_o$ be the cannonical inner-outer factorization of Θ. Since Θ_i is inner, Lemma 4.4 implies that the factorization is regular. Therefore, according to Theorem 3.2 (see also Theorem 3.3) and Theorem 3.6, the above factorization yields a triangulation

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{B}_i & 0 \\ * & \mathbf{A}_i \end{pmatrix}, \quad i = 1, \ldots, n,$$

of $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$, the functional model of T, such that the characteristic functions of $\mathbf{B} := [\mathbf{B}_1, \ldots, \mathbf{B}_n]$ and $\mathbf{A} := [\mathbf{A}_1, \ldots, \mathbf{A}_n]$ coincide with the purely contractive parts of Θ_i and Θ_o, respectively. Due to Lemma 3.4, the purely contractive part of an outer or inner multi-analytic operator is also outer or inner, respectively. We recall from [8] that a c.n.c. row contraction is of class $C_{\cdot 0}$ (resp. $C_{\cdot 1}$) if and only if the corresponding characteristic function is inner (resp. outer) multi-analytic operator. Finally, using the last part of Theorem 3.6, we can complete the proof. $\square$

5. Characteristic functions and joint similarity to Cuntz row isometries

In this section, we obtain criterions for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we prove that a completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$ is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator. This is a multivariable version of a result of Sz.-Nagy and Foiaş [24], concerning the similarity to unitary operators.

Extending on some results obtained by Sz.-Nagy [21], Nagy-Foiaş [25], and the author [6], [17], we provide necessary and sufficient conditions for a power bounded n-tuple of operators on a Hilbert space to be jointly similar to a Cuntz row isometry.

We need the following well-known result (see eg. [25]).

Lemma 5.1. *Let $\mathcal{M}, \mathcal{N}, \mathcal{X}$ and $\mathcal{Y}$ be subspaces of a Hilbert space $\mathcal{H}$ such that*

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N} = \mathcal{X} \oplus \mathcal{Y}.$$

If

$$P_{\mathcal{M}} \mathcal{X} = \mathcal{M} \quad \text{and} \quad \|P_{\mathcal{M}} x\| \geq c\|x\|, \quad x \in \mathcal{X},$$

for some constant $c > 0$, then

$$P_{\mathcal{N}} \mathcal{Y} = \mathcal{N} \quad \text{and} \quad \|P_{\mathcal{N}} y\| \geq c\|y\|, \quad y \in \mathcal{Y}.$$

We recall a few facts concerning the geometric structure of the minimal isometric dilation of a row contraction. Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. In [7], we proved that $\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*)$ and

$$P_{\mathcal{R}} h = \lim_{k \to \infty} \sum_{|\alpha| = k} V_\alpha T_\alpha^* h, \quad h \in \mathcal{H}, \tag{5.1}$$

where $P_{\mathcal{R}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}$. Moreover, if T is a one-to-one row contraction, then

$$\overline{P_{\mathcal{R}}\mathcal{H}} = \mathcal{R}. \tag{5.2}$$

The next result provides necessary and sufficient conditions for a c.n.c. row contraction to be jointly similar to a Cuntz row isometry, in terms of the corresponding characteristic function.

Theorem 5.2. *Let $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, be a completely non-coisometric row contraction. Then T is jointly similar to a Cuntz row isometry $W := [W_1, \dots, W_n]$, $W_i \in B(\mathcal{W})$, i.e.,*

(i) $W_1W_1^* + \cdots + W_nW_n^* = I_{\mathcal{W}}$;
(ii) $ST_i = W_iS$, $i = 1, \dots, n$, *for some invertible operator* $S : \mathcal{H} \to \mathcal{W}$,

if and only if the characteristic function Θ_T is an invertible multi-analytic operator.

In this case,

$$\|\Theta_T^{-1}\| = \min\left\{\|X\|\|X^{-1}\| : \ [X^{-1}T_1X, \dots, X^{-1}T_nX] \ \text{ is a Cuntz row isometry}\right\}.$$

Proof. Suppose that the row contraction $T := [T_1, \dots, T_n]$ is jointly similar to a Cuntz row isometry $W := [W_1, \dots, W_n]$, $W_i \in B(\mathcal{W})$, i.e.,

$$W_1W_1^* + \cdots + W_nW_n^* = I_{\mathcal{W}}$$

and $T_i = S^{-1}W_iS$, $i = 1, \dots, n$, for some invertible operator $S : \mathcal{H} \to \mathcal{W}$. Since $ST_\alpha = W_\alpha S$ and $T_\alpha^* S^* = S^* W_\alpha^*$ for any $\alpha \in \mathbb{F}_n^+$, we have

$$\begin{aligned} S\left(\sum_{|\alpha|=k} T_\alpha T_\alpha^*\right)S^* &= \sum_{|\alpha|=k} W_\alpha SS^* W_\alpha^* \\ &\geq \frac{1}{\|{S^*}^{-1}S^{-1}\|}\sum_{|\alpha|=k} W_\alpha W_\alpha^* \\ &= \frac{1}{\|S^{-1}\|^2} I \end{aligned}$$

for any $k = 1, 2, \dots$. Therefore,

$$\begin{aligned} \sum_{|\alpha|=k} \langle T_\alpha T_\alpha^* h, h\rangle &\geq \|{S^*}^{-1}h\|^2 \frac{1}{\|S^{-1}\|^2} \\ &\geq \frac{1}{\|S^*\|^2\|S^{-1}\|^2}\|h\|^2, \end{aligned}$$

which, due to relation (5.1), implies

$$\|P_{\mathcal{R}}h\| \geq \frac{1}{\|S\|\|S^{-1}\|}\|h\|, \quad h \in \mathcal{H}. \tag{5.3}$$

Notice that the operator $[T_1, \dots, T_n]$ is one-to-one. Indeed, the relation

$$S^{-1}W_1Sh_1 + \cdots + S^{-1}W_nSh_n = 0, \qquad h_i \in \mathcal{H}, \ i = 1, \dots, n,$$

implies

$$W_1Sh_1 + \cdots + W_nSh_n = 0.$$

Since W_i are isometries with orthogonal ranges, we have

$$W_iSh_i = 0, \qquad i = 1, \dots, n,$$

whence $h_i = 0$, $i = 1, \ldots, n$. Therefore $[T_1, \ldots, T_n]$ is one-to-one. According to (5.2), we have $\overline{P_{\mathcal{R}}\mathcal{H}} = \mathcal{R}$. Due to relation (5.3), the subspace $P_{\mathcal{R}}\mathcal{H}$ is closed. Therefore, $P_{\mathcal{R}}\mathcal{H} = \mathcal{R}$ and the operator

$$X := P_{\mathcal{R}}|_{\mathcal{H}} : \mathcal{H} \to \mathcal{R}$$

is invertible. According to (5.1), we have

$$\begin{aligned} V_i^* P_{\mathcal{R}} h &= \lim_{k\to\infty} \sum_{|\alpha|=k} V_i^* V_\alpha T_\alpha^* h \\ &= \lim_{k\to\infty} \sum_{|\alpha|=k-1} V_\beta T_\beta^* T_i^* h = P_{\mathcal{R}} T_i^* h \end{aligned}$$

for any $h \in \mathcal{H}$ and $i = 1, \ldots, n$. Consequently, we have

$$T_i X^* = X^* W_i, \qquad i = 1, \ldots, n,$$

where $W_i := V_i|_{\mathcal{R}}$, $i = 1, \ldots, n$. Due to the noncommutative Wold decomposition applied to the row isometry $[V_1, \ldots, V_n]$, the subspace $\mathcal{R}$ is reducing under each isometry V_i, $i = 1, \ldots,$ and $[W_1, \ldots, W_n]$ is a Cuntz row isometry.

Now, due to the geometric structure of the minimal isometric dilation of T , we have (see relation (2.2))

$$\mathcal{K} = \mathcal{R} \oplus M_V(\mathcal{L}_*) = \mathcal{H} \oplus M_V(\mathcal{L}).$$

Since $P_{\mathcal{R}}\mathcal{H} = \mathcal{R}$, we can use relation (5.3) and Lemma 5.1 to deduce that

$$P_{M_V(\mathcal{L}_*)} M_V(\mathcal{L}) = M_V(\mathcal{L}_*) \quad \text{ and } \quad \|P_{M_V(\mathcal{L}_*)} x\| \geq \frac{1}{\|S\|\|S^{-1}\|}\|x\|, \quad x \in M_V(\mathcal{L}).$$

Therefore, the operator

$$Q := P_{M_V(\mathcal{L}_*)}|_{M_V(\mathcal{L})} : M_V(\mathcal{L}) \to M_V(\mathcal{L}_*)$$

is an invertible contraction with $\|Q^{-1}\| \leq \|S\|\|S^{-1}\|$. Since Q is unitarily equivalent to the characteristic function Θ_T of T (see Section 2), we deduce that Θ_T is an invertible multi-analytic operator and $\|\Theta_T^{-1}\| \leq \|S\|\|S^{-1}\|$.

Conversely, assume that the characteristic function Θ_T (and hence Q) is an invertible contraction and $\|\Theta_T^{-1}\| \leq \frac{1}{c}$ for some constant $c > 0$. Applying again Lemma 5.1, we deduce that

$$P_{\mathcal{R}}\mathcal{H} = \mathcal{R} \quad \text{ and } \quad \|P_{\mathcal{R}} h\| \geq c\|h\|, \quad h \in \mathcal{H}.$$

This shows that the operator $X := P_{\mathcal{R}}|_{\mathcal{H}} : \mathcal{H} \to \mathcal{R}$ is invertible and $\|X^{-1}\| \leq \frac{1}{c}$. As in the first part of the proof, we have $X^*(V_i|_{\mathcal{R}}) = T_i X^*$ for any $i = 1, \ldots, n$. This proves the similarity to a Cuntz row isometry. Notice also that, since $\|X\| \leq 1$, we have

$$\|X^{*-1}\|\|X^*\| = \|X^{-1}\|\|X\| \leq \frac{1}{c}.$$

To prove the last part of the theorem, let $c > 0$ be such that $\|\Theta_T^{-1}\| = \frac{1}{c}$. The converse of this theorem implies the existence of on invertible operator X such that $[X^{-1}T_1X, \ldots, X^{-1}T_nX]$ is a Cuntz row isometry and

$$\|X\|\|X^{-1}\| \leq \frac{1}{c} = \|\Theta_T^{-1}\|.$$

On the other hand, using the first part of the proof, we have

$$\|\Theta_T^{-1}\| \leq \|X\|\|X^{-1}\|.$$

Therefore, $\|\Theta_T^{-1}\| = \|X\|\|X^{-1}\|$ and the proof is complete. □

Corollary 5.3. *If $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, is a completely non-coisometric row contraction jointly similar to a Cuntz row isometry, then T is jointly similar to the Cuntz part in the Wold decomposition of the minimal isometric dilation of T. Moreover, in this case, T is similar to the model row contraction $C := [C_1, \dots, C_n]$, where for each $i = 1, \dots, n$,*

$$C_i : \overline{\Delta_{\Theta_T}(F^2(H_n) \otimes \mathcal{D})} \to \overline{\Delta_{\Theta_T}(F^2(H_n) \otimes \mathcal{D})}$$

is defined by

$$C_i(\Delta_{\Theta_T} f) := \Delta_{\Theta_T}(S_i \otimes I_{\mathcal{D}})f, \quad f \in F^2(H_n) \otimes \mathcal{D},$$

and $\Delta_{\Theta_T} := (I - \Theta_T^* \Theta_T)^{1/2}$, where Θ_T is the characteristic function of T.

Proof. The first part of the theorem follows from the proof of Theorem 5.2. Now, using the model theory for c.n.c row contractions (see Theorem 2.1 and Theorem 2.2), one can complete the proof. □

Now we consider the case when $T := [T_1, \dots, T_n]$ is an arbitrary row contraction.

Theorem 5.4. *Let $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction. Then T is jointly similar to a Cuntz row isometry $W := [W_1, \dots, W_n]$, $W_i \in \mathcal{W}$, if and only if T is one-to-one and the operator*

$$P := \left(\text{SOT-}\lim_{k\to\infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right)^{1/2} \tag{5.4}$$

is invertible.

Moreover, if this is the case, then the row contraction $T := [T_1, \dots, T_n]$ is jointly similar to the Cuntz part $R := [R_1, \dots, R_n]$ in the Wold decomposition of the minimal isometric dilation of T.

Proof. Assume T is a similar to W, i.e., there exists an invertible operator $S : \mathcal{H} \to \mathcal{W}$ such that $T_i = S^{-1} W_i S$, $i = 1, \dots, n$. As in the proof of Theorem 5.2, one can show that the operator $[T_1, \dots, T_n]$ is one-to-one. According to (5.2), we have $\overline{P_{\mathcal{R}} \mathcal{H}} = \mathcal{R}$. On the other hand, due to relation (5.1), we deduce that

$$\|P_{\mathcal{R}} h\|^2 = \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|Ph\|^2, \qquad h \in \mathcal{H}, \tag{5.5}$$

where the operator P is well-defined by (5.4), due to the fact that $\left\{ \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right\}_{k=1}^\infty$ is a decreasing sequence of positive operators. Notice that, since $\{W_\alpha\}_{|\alpha|=k}$ are isometries with orthogonal ranges, we have

$$\begin{aligned}\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 &\geq \|S^{-1}\|^{-2} \sum_{|\alpha|=k} \|W_\alpha^* S^{*-1} h\|^2 \\ &= \|S^{-1}\|^{-2} \|S^{*-1} h\|^2 \geq (\|S^{-1}\|^2 \|S\|^2)^{-1} \|h\|^2\end{aligned}$$

for any $h \in \mathcal{H}$. Therefore

$$\|P_{\mathcal{R}} h\|^2 = \|Ph\|^2 \geq (\|S^{-1}\|^2 \|S\|^2)^{-1} \|h\|^2$$

for any $h \in \mathcal{H}$. Hence, it follows that the operators P and $P_{\mathcal{R}}|_{\mathcal{H}}$ are one-to-one and have closed ranges. Since $\overline{P_{\mathcal{R}} \mathcal{H}} = \mathcal{R}$, it is clear that the operator $X : \mathcal{H} \to \mathcal{R}$ is invertible.

According to relation (5.1), we have

$$V_i^* P_{\mathcal{R}} h = \lim_{k\to\infty} \sum_{|\alpha|=k-1} V_\beta T_\beta^* T_i^* h = P_{\mathcal{R}} T_i^* h$$

for any $h \in \mathcal{H}$ and $i = 1, \dots, n$. Consequently, we deduce that

$$XT_i^* = R_i^* X, \qquad i = 1, \dots, n, \tag{5.6}$$

where $X := P_{\mathcal{R}}|_{\mathcal{H}}$ and $R_i := V_i|_{\mathcal{R}}$, $i = 1, \dots, n$. Therefore, $T := [T_1, \dots, T_n]$ is jointly similar to $R := [R_1, \dots, R_n]$.

Conversely, assume that the row contraction $[T_1, \dots, T_n]$ is one-to-one and the operator P is invertible. Then relation (5.5) implies $P_{\mathcal{R}}|_{\mathcal{H}}$ is one-to-one and has closed range. On the other hand, by (5.2), we have $\overline{P_{\mathcal{R}}\mathcal{H}} = \mathcal{R}$. Therefore, the operator $X := P_{\mathcal{R}}|_{\mathcal{H}} : \mathcal{H} \to \mathcal{R}$ is invertible and, due to relation (5.6), the row contraction $[T_1, \dots, T_n]$ is jointly similar to the Cuntz row isometry $[V_1|_{\mathcal{R}}, \dots, V_n|_{\mathcal{R}}]$. The proof is complete. □

We recall ([17]) that an n-tuple $[T_1, \dots, T_n]$, of operators $T_i \in B(\mathcal{H})$, is power bounded if there is a constant $M > 0$ such that

$$\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \le M^2 \|h\|^2, \quad h \in \mathcal{H},$$

for any $k = 1, 2, \dots$.

Theorem 5.5. *Let $[T_1, \dots, T_n]$ be a one-to-one power bounded n-tuple of operators on a Hilbert space $\mathcal{H}$ such that, for any non-zero element $h \in \mathcal{H}$, $\sum_{|\alpha|=k} \|T_\alpha^* h\|^2$ does not converges to 0 as $k \to \infty$. Then there exists a Cuntz row isometry $[W_1, \dots, W_n]$, $W_i \in B(\mathcal{H})$, such that*

$$T_i X = X W_i, \quad i = 1, \dots, n,$$

for some one-to-one operator $X \in B(\mathcal{H})$ with range dense in $\mathcal{H}$.

Proof. For each $h \in \mathcal{H}$, $h \ne 0$, denote

$$c(h) := \inf_{k=1,2,\dots} \left(\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \right)^{1/2}.$$

Since $[T_1, \dots, T_n]$ is a power bounded n-tuple of operators, there is aconstant $M > 0$ such that

$$\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \le M^2 \|h\|^2, \quad h \in \mathcal{H}, \tag{5.7}$$

for any $k = 1, 2, \dots$. If $c(h) = 0$ and $\epsilon > 0$, then there is k_0 such that

$$\left(\sum_{|\alpha|=k_0} \|T_\alpha^* h\|^2 \right)^{1/2} \le \frac{\epsilon}{M}.$$

Hence and using (5.7), we deduce that

$$\begin{aligned}\sum_{|\alpha|=m+k_0} \|T_\alpha^* h\|^2 &= \sum_{|\beta|=k_0} \left\langle T_\beta \left(\sum_{|\gamma|=m} T_\gamma T_\gamma^* \right) T_\beta^* h, h \right\rangle \\ &\le M^2 \sum_{|\beta|=k_0} \left\langle T_\beta T_\beta^* h, h \right\rangle \le \epsilon^2\end{aligned}$$

for any $m \geq 0$. Consequently, $\lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = 0$, which contradicts the hypothesis. Therefore, we must have $c(h) \neq 0$ for any $h \in \mathcal{H}$, $h \neq 0$.

Now, for each $h, h' \in \mathcal{H}$, we define

$$[h, h'] := \operatorname*{LIM}_{k\to\infty} \sum_{|\alpha|=k} \left\langle T_\alpha^* h, T_\alpha^* h' \right\rangle,$$

where LIM is a Banach limit. Due to the properties of the Banach limit, $[\cdot,\cdot]$ is a bilinear form on $\mathcal{H}$ and we deduce that

$$[h, h] := \operatorname*{LIM}_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \geq c(h)^2 > 0 \quad \text{ if } h \in \mathcal{H},\ h \neq 0,$$

and $[h, h] \leq M^2\|h\|^2$. Moreover, we have

$$[h, h] = \sum_{i=1}^n [T_i^* h, T_i^* h], \quad h \in \mathcal{H}.$$

Due to a well-known theorem on bounded hermitian forms, there exists a self-adjoint operator $P \in B(\mathcal{H})$ such that

$$[h, h'] = \left\langle Ph, h' \right\rangle \quad \text{ for any } h, h' \in \mathcal{H},$$

and, due to the above considerations, we have

$$0 < \langle Ph, h\rangle < M^2\|h\|^2, \quad h \in \mathcal{H},\ h \neq 0. \tag{5.8}$$

Now, we show that $P = \sum_{i=1}^n T_i P T_i^*$. Indeed, we have

$$\begin{aligned}
\langle Ph, h\rangle &= \operatorname*{LIM}_{k\to\infty} \sum_{|\alpha|=k+1} \|T_\alpha^* h\|^2 = \operatorname*{LIM}_{k\to\infty} \sum_{i=1}^n \sum_{|\alpha|=k} \|T_\alpha^* T_i^* h\|^2 \\
&= \sum_{i=1}^n [T_i^* h, T_i^* h] = \sum_{i=1}^n \langle P T_i^* h, T_i^* h\rangle \\
&= \sum_{i=1}^n \left\langle \sum_{i=1}^n T_i P T_i^* h, h \right\rangle
\end{aligned}$$

for any $h \in \mathcal{H}$, which proves our assertion. Notice that relation (5.8) shows that the operator $X := P^{1/2}$ is one-to-one and has range dense in $\mathcal{H}$. Since $\sum_{i=1}^n \|XT_i^* h\|^2 = \|Xh\|^2$ for any $h \in \mathcal{H}$, it is clear that

$$\sum_{i=1}^n \|XT_i^* X^{-1} x\|^2 = \|x\|^2$$

for any x in the domain on X^{-1}. Hence and due to the fact that the domain on X^{-1} is dense in $\mathcal{H}$, the operators $V_i^* := XT_i^* X^{-1}$, $i = 1, \dots, n$, can be extended by continuity on $\mathcal{H}$. Using the same notation for the corresponding extensions, we have

$$\sum_{i=1}^n \|V_i^* h\|^2 = \|h\|^2, \quad h \in \mathcal{H},$$

and $V_i^* X = XT_i^*$, $i = 1, \dots, n$. This shows that $[V_1, \dots, V_n]$ is a co-isometry from $\mathcal{H}^{(n)}$ to $\mathcal{H}$ such that

$$T_i X = XV_i, \quad i = 1, \dots, n.$$

Assume now that $h_i \in \mathcal{H}$ and $\sum_{i=1}^n V_i h_i = 0$. Then $\sum_{i=1}^n T_i X h_i = 0$. Since $[T_1, \dots, T_n]$ and X are one-to-one operators, we must have $h_i = 0$ for each $i = 1, \dots, n$. Consequently, $[V_1, \dots, V_n]$ is a one-to-one co-isometry, and therefore a unitary operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}$. This implies that $V_1, \dots, V_n$ are isometries on $\mathcal{H}$ with $V_1 V_1^* + \cdots + V_n V_n^* = I_{\mathcal{H}}$. The proof is complete. □

As a consequence of Theorem 5.5, we deduce the following criterion for joint similarity of a power bounded n-tuple of operators to a Cuntz row isometry.

Corollary 5.6. *Let $[T_1, \dots, T_n]$ be a one-to-one power bounded n-tuple of operators on a Hilbert space $\mathcal{H}$. Then $[T_1, \dots, T_n]$ is jointly similar to a Cuntz row isometry if and only if there exists a constant $c > 0$ such that*

$$\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \geq c\|h\|^2, \quad h \in \mathcal{H}, \tag{5.9}$$

for any $k = 1, 2, \dots$.

Proof. The direct implication can be extracted from the proof of Theorem 5.2. Conversely, if condition (5.9) holds, then, using the proof of Theorem 5.5, we have

$$c(h) \geq \sqrt{c}\|h\|, \quad h \in \mathcal{H},\ h \neq 0.$$

Moreover, the positive operator $P \in B(\mathcal{H})$ has the properties

$$T_i P^{1/2} = P^{1/2} V_i, \quad i = 1, \dots, n,$$

where $[V_1, \dots, V_n]$ is a Cuntz isometry, and

$$\langle Ph, h \rangle \geq c\|h\|^2, \quad h \in \mathcal{H},\ h \neq 0.$$

Since the latter inequality shows that $P^{1/2}$ is an invertible operator, the result follows. □

References

[1] W.B. Arveson, The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \dots, z_n]$, *J. Reine Angew. Math.* **522** (2000), 173–236.
[2] J. W. Bunce, Models for n-tuples of noncommuting operators, *J. Funct. Anal.* **57** (1984), 21–30.
[3] A. E. Frazho, Models for noncommuting operators, *J. Funct. Anal.* **48** (1982), 1–11.
[4] D. Kribs, The curvature invariant of a non-commuting N-tuple, *Integral Equations Operator Theory* **41** (2001), no. 4, 426–454.
[5] G. Pisier, *Similarity Problems and Completely Bounded Maps*, Springer Lect. Notes Math., Vol.1618, Springer-Verlag, New York, 1995.
[6] G. Popescu, Models for infinite sequences of noncommuting operators, *Acta. Sci. Math. (Szeged)* **53** (1989), 355–368.
[7] G. Popescu, Isometric dilations for infinite sequences of noncommuting operators, *Trans. Amer. Math. Soc.* **316** (1989), 523–536.
[8] G. Popescu, Characteristic functions for infinite sequences of noncommuting operators, *J. Operator Theory* **22** (1989), 51–71.
[9] G. Popescu, Multi-analytic operators and some factorization theorems, *Indiana Univ. Math. J.* **38** (1989), 693–710.
[10] G. Popescu, Von Neumann inequality for $(B(H)^n)_1$, *Math. Scand.* **68** (1991), 292–304.
[11] G. Popescu, On intertwining dilations for sequences of noncommuting operators, *J. Math. Anal. Appl.* **167** (1992), 382–402.
[12] G. Popescu, Functional calculus for noncommuting operators, *Michigan Math. J.* **42** (1995), 345–356.
[13] G. Popescu, Multi-analytic operators on Fock spaces, *Math. Ann.* **303** (1995), 31–46.
[14] G. Popescu, Poisson transforms on some C^*-algebras generated by isometries, *J. Funct. Anal.* **161** (1999), 27–61.
[15] G. Popescu, Curvature invariant for Hilbert modules over free semigroup algebras, *Adv. Math.* **158** (2001), 264–309.

[16] G. Popescu, Central intertwining lifting, suboptimization, and interpolation in several variables, *J. Funct. Anal.* **189** (2002), 132–154.
[17] G. Popescu, Similarity and ergodic theory of positive linear maps, *J. Reine Angew. Math.*, **561** (2003), 87–129.
[18] G. Popescu, Entropy and multivariable interpolation, *Mem. Amer. Math. Soc.*, to appear.
[19] G. Popescu, Operator theory on noncommutative varieties, *Indiana Univ. Math. J.*, to appear.
[20] G. Popescu, Operator theory on noncommutative varieties II, preprint.
[21] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space *Acta. Sci. Math. (Szeged)* **11** (1947), 152–157.
[22] B. Sz.-Nagy and C. Foiaş, Une caractérisation de sous-espaces invariants pour une contraction de l'espace de Hilbert, *C.R. Acad. Sci. Paris*, **258** (1964), 3426-3429.
[23] B. Sz.-Nagy and C. Foiaş, Sur les contractions de l'espace de Hilbert. IX. Factorisations de la fonction caractéristique. Sous-espaces invariants, *Acta. Sci. Math. (Szeged)* **25** (1964), 283–316.
[24] B. Sz.-Nagy and C. Foiaş, Sur les contractions de l'espace de Hilbert. X. Contractions similaires à des transformations unitaires, *Acta. Sci. Math. (Szeged)* **26** (1965), 79–91.
[25] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York 1970.
[26] J. von Neumann, Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.* **4** (1951), 258–281.

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